AP Calculus Notes: Unit 5 – Applications of Derivatives

TEACHER NOTE: This appears to be a short unit, with only two objectives. However, students will need much more practice on these two topics. Be sure to give additional time and practice problems in which students must solve a variety of optimization and related rates problems mixed together.

Syllabus Objective: 2.12 – The student will solve optimization problems.

Optimization (Max/Min) Problem: a problem in which a quantity is to be maximized or minimized

Steps to Solving a Max/Min Problem

1. Assign symbols to all given quantities and quantities to be determined. Make a sketch.
2. Write a PRIMARY EQUATION for the quantity being maximized or minimized.
3. Reduce the primary equation to one having a single independent variable.
4. Determine the maximum/minimum using critical values.
5. Use the first (or second) derivative test and choose the feasible answer.
6. Be sure to answer the question asked. Include units with your answer (if applicable).

Teacher Note: These problems MUST be solved using Calculus on the AP exam. No credit will be earned for precalculus methods.

Ex 1: Find two numbers whose sum is 16 and whose product is as large as possible.

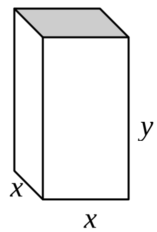
Using the steps outlined above:

1. Two numbers: x, y; $y = 16 - x$
2. Product (maximize): $P = xy$
3. $y = 16 - x$: $P = x(16 - x)$
4. $P = x(16-x) = 16x - x^2$ $\quad P' = 16 - 2x \quad 16 - 2x = 0 \Rightarrow x = 8$
5. Second derivative test: $P'' = -2 \Rightarrow P''(8) = -2 < 0 \therefore 8$ is a local max
6. $x = 8,\ y = 16 - 8 = 8$ $\quad$ Solution: $\boxed{8, 8}$

Ex 2: A manufacturer wants to design an open box having a square base and a surface area of 147 in^2. What dimensions will produce a box with maximum volume?

Using the steps outlined above:

1.



2. Volume of Box (maximize): $V = x^2 y$

3. Surface Area: $147 = x^2 + 4xy \Rightarrow y = \dfrac{147}{x} - x$ $\qquad V = x^2\left(\dfrac{147}{x} - x\right) = 147x - x^3$

4. $V' = 147 - 3x^2 = 0 \Rightarrow x^2 = 49 \Rightarrow x = \pm 7$

5. Second derivative test: $V'' = -6x \Rightarrow P''(7) = -42 < 0 \;\therefore\; 7$ is a local max

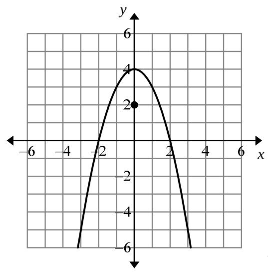
   Note: $x = -7$ is not feasible for the length of a side.

6. $y = \dfrac{147}{x} - x \Rightarrow y = \dfrac{147}{7} - 7 = 14$ Solution: $\boxed{7\,\text{in} \times 7\,\text{in} \times 14\,\text{in}}$

Ex 3: Which points on the graph of $y = 4 - x^2$ are closest to the point $(0,2)$?

Using the steps outlined above:

1.



Point on the graph: (x, y)

2. Distance from $(0,2)$ to (x, y) (minimize): $d = \sqrt{(x-0)^2 + (y-2)^2}$

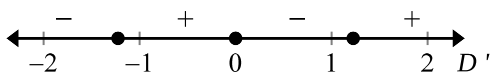
3. $y = 4 - x^2$: $\quad d = \sqrt{(x-0)^2 + (4 - x^2 - 2)^2} = \sqrt{x^2 + (2 - x^2)^2}$

4. If you are minimizing a square root, it is sufficient to minimize the value under the radical.

   We will call this value *D*. $\quad D = x^2 + (2 - x^2)^2$

   $D' = 2x + 2(2 - x^2)(-2x) = 4x^3 - 6x = 0 \Rightarrow 2x(2x^2 - 3) = 0, \Rightarrow x = 0, \pm\sqrt{\frac{3}{2}}$

5. Sign Chart:



   $x = \pm\sqrt{\frac{3}{2}}$ are local minima because D' changes sign from negative to positive.

6. $y = 4 - \left(-\sqrt{\frac{3}{2}}\right)^2 = \frac{5}{2}$, $y = 4 - \left(\sqrt{\frac{3}{2}}\right)^2 = \frac{5}{2}$ $\quad$ Solutions: $\boxed{\left(-\sqrt{\frac{3}{2}}, \frac{5}{2}\right), \left(\sqrt{\frac{3}{2}}, \frac{5}{2}\right)}$

<u>You Try:</u> A rectangular page is to contain 24 in^2 of print. The margins at the top and bottom of the page are to be $1\frac{1}{2}$ inches, and the margins to the left and right are to be 1 inch. What should the dimensions of the page be so that the least amount of paper is used?

<u>QOD:</u> Give a real-life example (other than the ones given in the notes) of something that would need to be maximized or minimized.

Syllabus Objective: 2.14 – The student will solve problems involving rates of change.

Related Rates Equation: an equation that relates the corresponding rates of two or more variables that are differentiable functions of time *t*

<u>Steps to Solving a Related Rates Problem</u>

1. Make a sketch (if possible). Name the variables and constants.

2. Write down the known information and the variable we are to find.

3. Write an equation that relates the variables.

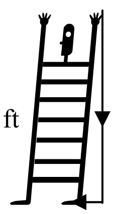
4. Differentiate implicitly the equation with respect to *t* using the chain rule.

   A common mistake is forgetting the chain rule. Be sure to use $\frac{dy}{dt}$ notation for derivatives, NOT y'.

5. Answer the question that was asked with correct units.

Ex 1: A 13-ft ladder is leaning (flush) against a wall. Suppose that the base of the ladder slides away from the wall at the constant rate of 3 ft/sec. Find the rate at which the top of the ladder is moving down the wall at $t=1$ sec.

Using the steps outlined above:



1. *x* & *y* are **variables** because they are changing. The length of the ladder is **constant**.

2. $t=1\text{ sec},\ \frac{dx}{dt}=3\ {}^{\text{ft}}/_{\text{sec}} \Rightarrow x=3\text{ft}$

Note: $\frac{dx}{dt}$ is positive because *x* is increasing. Choose the sign of each rate wisely!

$\frac{dy}{dt}=?$ Note: $\frac{dy}{dt}$ will be negative because *y* is decreasing.

3. Equation (Pythagorean Theorem): $x^2+y^2=13^2$

4. Differentiate with respect to *t*: $2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$

5. In order to find $\frac{dy}{dt}$, we also need *y*. Use the equation from Step 3: $3^2+y^2=13^2 \Rightarrow y=\sqrt{160}$

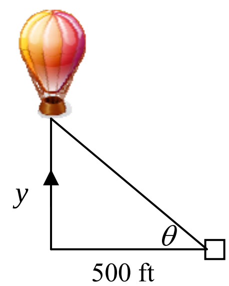
$$2(3)(3)+2\left(\sqrt{160}\right)\frac{dy}{dt}=0 \Rightarrow 8\sqrt{10}\frac{dy}{dt}=-18 \Rightarrow \frac{dy}{dt}=-\frac{9}{4\sqrt{10}}\approx -0.712$$

Solution: Since the question asked the rate the ladder was moving **down** the wall: $\boxed{0.712\ {}^{\text{ft}}/_{\text{sec}}}$

Ex 2: A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Using the steps outlined above:

1.



Variables: $\theta,\ y$ Constant: 500 ft

2. $\theta = \frac{\pi}{4},\ \frac{d\theta}{dt} = 0.14\ ;\ \frac{dy}{dt} = ?$

3. Equation (trig): $\tan\theta = \frac{y}{500}$

4. Differentiate with respect to *t*: $\sec^2\theta \frac{d\theta}{dt} = \frac{1}{500}\frac{dy}{dt}$

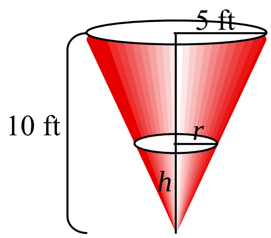
5. $\sec^2\frac{\pi}{4}(0.14) = \frac{1}{500}\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = 140$

6. Solution: $\boxed{140\ \text{ft}/\text{sec}}$

Ex 3: Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

Using the steps outlined above:

1.



Variables: r, h; Note the relationship: $\frac{r}{h}=\frac{5}{10}$ or $10r=5h \Rightarrow r=\frac{1}{2}h$

2. $\frac{dV}{dt}=9\,\text{ft}^3/\text{min}$, $h=6$ ft, $\frac{dh}{dt}=?$

3. Equation (volume): $V=\frac{\pi}{3}r^2h$; Note that we can substitute $r=\frac{1}{2}h$: $V=\frac{\pi}{3}\left(\frac{1}{2}h\right)^2 h=\frac{\pi}{12}h^3$

4. Differentiate with respect to t: $V=\frac{\pi}{12}h^3 \Rightarrow \frac{dV}{dt}=\frac{\pi}{4}h^2\frac{dh}{dt}$

5. $9=\frac{\pi}{4}(6)^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt}=\frac{1}{\pi}$

6. Solution: $\boxed{\frac{1}{\pi}\ \text{ft/sec}}$

Note: When determining the units for the answer, use the units from the original problem. For example, if you are determining units for $\frac{dh}{dt}$, it would be the units for h (ft) over the units for t (sec).

<u>You Try:</u> A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 miles north of the intersection and the car is 0.8 miles to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

<u>QOD:</u> Explain why you must differentiate with respect to t when solving a related rates problem.

Sample AP Calculus AB Exam Question(s) (taken from the released 2003 MC AP Exam):

1. The radius of a circle is increasing at a constant rate of 0.2 meters per second. What is the rate of increase in the area of the circle at the instant when the circumference of the circle is 20π meters?

   (A) $0.04\pi \text{ m}^2/\text{sec}$

   (B) $0.4\pi \text{ m}^2/\text{sec}$

   (C) $4\pi \text{ m}^2/\text{sec}$

   (D) $20\pi \text{ m}^2/\text{sec}$

   (E) $100\pi \text{ m}^2/\text{sec}$

2. The rate of change of the volume, V, of water in a tank with respect to time, t, is directly proportional to the square root of the volume. Which of the following is a differential equation that describes this relationship?

   (A) $V(t) = k\sqrt{t}$

   (B) $V(t) = k\sqrt{V}$

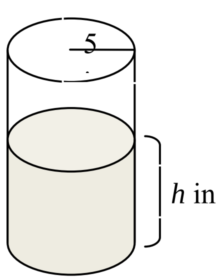
   (C) $\dfrac{dV}{dt} = k\sqrt{t}$

   (D) $\dfrac{dV}{dt} = \dfrac{k}{\sqrt{V}}$

   (E) $\dfrac{dV}{dt} = k\sqrt{V}$

Free Response Question

A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure. Let *h* be the depth of the coffee in the pot, measured in inches, where *h* is a function of time *t*, measured in seconds. The volume *V* of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume *V* of a cylinder with radius *r* and height *h* is $V = \pi r^2 h$.)



Show that $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$.