AP Calculus Notes: Unit 4 – Derivatives & Graphing

Syllabus Objective: 2.11 – The student will sketch curves using derivatives and limits.

Absolute (Global) Extreme Values of Functions (Absolute Extrema)

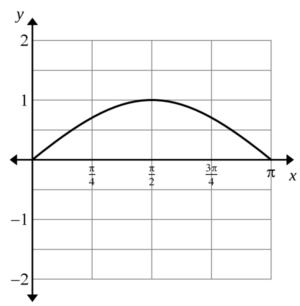
Absolute (Global) Maximum: the value $f(c)$, if $f(c) \geq f(x)$ for all x in the domain of $f(x)$

Absolute (Global) Minimum: the value $f(c)$, if $f(c) \leq f(x)$ for all x in the domain of $f(x)$

Note: Absolute Extreme Values are sometimes referred to as simply Maximum or Minimum Values of a function.

Extreme Value Theorem: A continuous function has both an absolute maximum and absolute minimum on a closed interval.

Ex1: Find the absolute extrema of $f(x) = \sin x$ on the interval $[0, \pi]$.



Looking at the graph, we can see that the absolute **maximum** value of $f(x) = \sin x$ is $\boxed{1}$. Note: This occurs when $x = \frac{\pi}{2}$.

Also from the graph, we find that the absolute **minimum** value of $f(x) = \sin x$ is $\boxed{0}$. Note: This occurs when $x = 0$, and when $x = \pi$.

Teacher Note: Have students discuss the absolute extrema of $f(x) = \sin x$ over its entire domain $(-\infty, \infty)$. Emphasize that the extreme values can differ when the domain restrictions are changed.

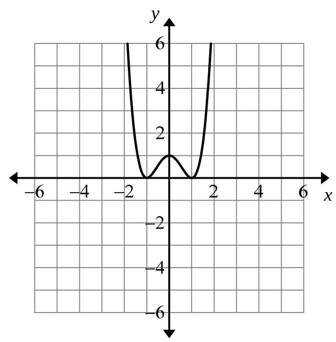
Local (Relative) Extreme Values of Functions

Local (Relative) Maximum: the value $f(c)$, if $f(c) \geq f(x)$ for all x in an open interval containing c

Local (Relative) Minimum: the value $f(c)$, if $f(c) \leq f(x)$ for all x in an open interval containing c

Note: Local extrema are the values of a function that are the smallest or largest in a *nearby* location (in the neighborhood). A function could have multiple relative minima and/or relative maxima.

Ex2: Find any local and absolute extrema of the function shown on its entire domain.



$f(x) = x^4 - 2x^2 + 1$

Local Minimum: The local minimum is $\boxed{0}$. This occurs when $x = -1$ and when $x = 1$.

Local Maximum: The local maximum is $\boxed{1}$. This occurs when $x = 0$.

Absolute Minimum: The absolute minimum is $\boxed{0}$. This occurs when $x = -1$ and when $x = 1$.

Absolute Maximum: The function does **not** have an absolute maximum on its domain $(-\infty, \infty)$. It increases without bound.

Theorem – Local Extreme Values: If a function *f* has a local maximum value or a local minimum value at an interior point *c* of its domain, then either $f'(c) = 0$ or $f'(c)$ **does not exist**.
Teacher Note: Allow students time to discuss why this is true and have them give examples of each case.

Critical Point: a point in the interior of the domain of a function *f* at which $f'(c) = 0$ or $f'(c)$ does not exist.
Note: Extreme values can **only** occur at critical points and endpoints.

Ex3: Find the extrema of $f(x) = \frac{1}{x} + \ln x$ on the closed interval on the closed interval $\left[\frac{1}{2}, 4\right]$.

Critical Point(s): $f'(x) = -\frac{1}{x^2} + \frac{1}{x}$ Find the values of *x* where $f'(x) = 0$ or does not exist.

$f'(x) = \frac{-1+x}{x^2} = 0 \Rightarrow -1 + x = 0 \Rightarrow x = 1$ $\quad f(1) = \frac{1}{1} + \ln 1 = 1$ $\quad$ Critical Point: $(1,1)$

$f'(x) = \frac{-1+x}{x^2}$ does not exist when $x^2 = 0 \Rightarrow x = 0$ $\quad$ $x = 0$ is not in the domain

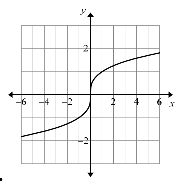
Endpoints: $f\left(\frac{1}{2}\right) = 2 + \ln\left(\frac{1}{2}\right) \approx 1.307$ $\quad f(4) = 2 + \ln(4) \approx 3.386$ $\quad \left(\frac{1}{2}, 1.307\right), (4, 3.386)$

Absolute Minimum = $\boxed{1}$ $\quad$ Absolute Maximum = $\boxed{3.386}$

<u>Note:</u> A critical point is *not necessarily* an extreme value. For example, the function $y = x^3$ has a critical point at $(0,0)$ because $f'(0) = 0$. However, $f(0) = 0$ is **not** an extreme value.

Ex4: Find the critical point(s) of the function $f(x) = x^{\frac{1}{3}}$. Then find any extreme values (local and absolute).

$f'(x) = \dfrac{1}{3x^{\frac{2}{3}}}$ $\qquad f'(x) \neq 0$ $\qquad f'(x)$ does not exist when $x = 0$ $\qquad$ Critical Point: $\boxed{(0,0)}$



$f(x) = x^{\frac{1}{3}}$ has no extreme values.

<u>You Try:</u> Identify the critical point(s) and determine the local extreme values of $f(x) = \begin{cases} x^2 & x \leq 1 \\ 3x - 2 & x > 1 \end{cases}$.

<u>QOD:</u> Give an example of a function that has only an absolute maximum value (not an absolute minimum) on its entire domain. Give an example of a function that has only an absolute minimum value (not an absolute maximum) on its entire domain. Then give an example of a function that does not have any absolute extrema on its entire domain.

<u>Sample AP Calculus AB Exam Question(s) (taken from the released 2003 MC AP Exam):</u>

Let f be the function with derivative given by $f'(x) = \sin(x^2 + 1)$. How many relative extrema does f have on the interval $2 < x < 4$?

(A) One

(B) Two

(C) Three

(D) Four

(E) Five

Syllabus Objective: 2.9 – The student will interpret graphically the Mean Value Theorem.

Mean Value Theorem (Derivatives): Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). There is at least on point c in (a,b) at which $f'(c)=\frac{f(b)-f(a)}{b-a}$.

Note: This means that within the interval there must be a tangent line with the same slope as the secant line that passes through the endpoints of the interval.

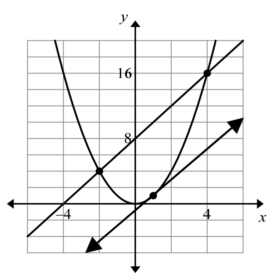
Ex1: Show that the function $f(x)=x^2$ satisfies the hypotheses of the mean value theorem over the interval $[-2,4]$. Then find a value of c such that $f'(c)=\frac{f(b)-f(a)}{b-a}$ on this interval. Graph the secant and tangent lines to verify your answer.

$f(x)=x^2$ is continuous on the interval $[-2,4]$ and differentiable on the interval $(-2,4)$. Therefore, it satisifies the conditions of the Mean Value Theorem.

$$f'(x)=2x \qquad \frac{f(b)-f(a)}{b-a}=\frac{f(4)-f(-2)}{4-(-2)}=\frac{16-4}{6}=2$$

The Mean Value Theorem states that there must be a value of c such that $f'(c)=2$, or $2c=2 \Rightarrow \boxed{c=1}$.

Graph:



Physical Interpretation of the Mean Value Theorem: At some point in a given interval, the instantaneous rate of change must equal the average rate of change.

Ex2: Suppose it took you 5 hours to drive the 330-mile trip to San Diego, CA. Calculate your average speed and explain why your speed was equal to this average speed at some time during the 5 hours.

Average Speed = $\frac{\text{change in distance}}{\text{change in time}}=\frac{330}{5}=66\ {}^{\text{mi}}\!/\!_{\text{hr}}$

Because the function representing your distance over time is continuous and differentiable, it satisfies the Mean Value Theorem. Therefore, at some point during the 5 hours, your speed was equal to exactly $66\ {}^{\text{mi}}\!/\!_{\text{hr}}$.

Increasing Function: A function is increasing on an interval *I* if for all $x_1 < x_2$ in *I*, $f(x_1) < f(x_2)$.

Let f be continuous on $[a,b]$ and differentiable on (a,b). f increases on $[a,b]$ if $f' > 0$.

Decreasing Function: A function is increasing on an interval *I* if for all $x_1 < x_2$ in *I*, $f(x_1) > f(x_2)$.

Let f be continuous on $[a,b]$ and differentiable on (a,b). f decreases on $[a,b]$ if $f' < 0$.

Monotonic Function: a function that is always increasing or always decreasing on an interval

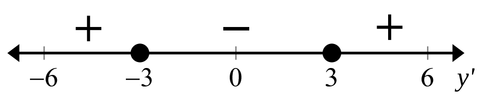
Constant Function: a function that can be defined as $f(x) = C$, where *C* is a constant, and $f'(x) = 0$

Ex3: Determine what intervals the function $f(x) = x^3 - 27x$ is increasing and decreasing (using calculus).

To determine which intervals the function is increasing or decreasing, we must find where the derivative is positive and negative. So we will first determine where the derivative is equal to 0 or does not exist (critical points).

$f'(x) = 3x^2 - 27$ $\qquad$ $3x^2 - 27 = 0 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$: critical values

Make a sign chart. Use test values to the left and right of the critical values to determine where the first derivative is positive and negative.



Because $f'(x) > 0$, the function $f(x) = x^3 - 27x$ is **increasing** on the intervals $\boxed{(-\infty,-3] \text{ and } [3,\infty)}$.

Because $f'(x) < 0$, the function $f(x) = x^3 - 27x$ is **decreasing** on the interval $\boxed{[-3,3]}$.

Check your answer by looking at the graph of $f(x) = x^3 - 27x$.

> Note: It is debatable whether to use closed or open intervals to describe increasing/decreasing behavior. The College Board does not take a stance on this issue and accepts either way as correct.

Antiderivative: a function $F(x)$ for which $F'(x) = f(x)$ for all *x* in the domain of *f*

<u>Note:</u> If two functions have the same derivative, then the functions differ by a constant. So, if $f'(x) = g'(x)$, then $f(x) = g(x) + C$ because the derivative of a constant is equal to 0.

Ex4: Find a function $f(x)$ with the derivative of $\cos x$ that passes through the point $\left(\frac{\pi}{2},3\right)$.

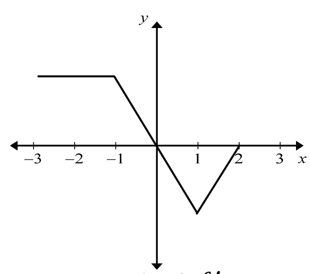
A function with the derivative of $\cos x$ is $\sin x$. So $f(x)=\sin x+C$ represents all functions with the derivative of $\cos x$. Use the given point to determine the particular function.

$$f(x)=\sin x+C \Rightarrow 3=\sin\left(\frac{\pi}{2}\right)+C \Rightarrow 3=1+C \Rightarrow 2=C \qquad \text{Solution: } \boxed{f(x)=\sin x+2}$$

<u>You Try:</u> Find the intervals on which the function $f(x)=4-\sqrt{x+2}$ is increasing and decreasing using calculus. Verify your answer by graphing.

<u>QOD:</u> Explain why monotonic functions are one-to-one functions.

<u>Sample AP Calculus AB Exam Question(s) (taken from the released 2003 MC AP Exam):</u>



Graph of f'

1. The graph of f', the derivative of the function f, is shown above. Which of the following statements is true about f?

   (A) f is decreasing for $-1 \le x \le 1$.

   (B) f is increasing for $-2 \le x \le 0$.

   (C) f is increasing for $1 \le x \le 2$.

   (D) f has a local minimum at $x=0$.

   (E) f is not differentiable at $x=-1$ and $x=1$.

2. Let f be the function with derivative given by $f'(x) = x^2 - \frac{2}{x}$. On which of the following intervals is f decreasing?

   (A) $(-\infty, -1]$ only

   (B) $(-\infty, 0)$

   (C) $[-1, 0)$ only

   (D) $(0, \sqrt[3]{2}]$

   (E) $[\sqrt[3]{2}, \infty)$

x	-4	-3	-2	-1	0	1	2	3	4
$g'(x)$	2	3	0	-3	-2	-1	0	3	2

3. The derivative g' of a function g is continuous and has exactly two zeros. Selected values of g' are given in the table above. If the domain of g is the set of all real numbers, then g is decreasing on which of the following intervals?

   (A) $-2 \le x \le 2$ only

   (B) $-1 \le x \le 1$ only

   (C) $x \ge -2$

   (D) $x \ge 2$ only

   (E) $x \le -2$ or $x \ge 2$

4. A curve has slope $2x + 3$ at each point (x, y) on the curve. Which of the following is an equation for this curve if it passes through the point $(1, 2)$?

   (A) $y = 5x - 3$

   (B) $y = x^2 + 1$

   (C) $y = x^2 + 3x$

   (D) $y = x^2 + 3x - 2$

   (E) $y = 2x^2 + 3x - 3$

Syllabus Objective: 2.11 – The student will sketch curves using derivatives and limits.

<u>First Derivative Test:</u> (at a critical point *c*)

1. If f' changes sign from positive to negative at *c*, then f has a local **maximum** value at *c*.
   $f' > 0$ for $x < c$ and $f' < 0$ for $x > c$
2. If f' changes sign from negative to positive at *c*, then f has a local **minimum** value at *c*.
   $f' < 0$ for $x < c$ and $f' > 0$ for $x > c$

Note: A critical point does not guarantee a local maximum or minimum. There must be a **sign change** at the critical point.

<u>Endpoints:</u>

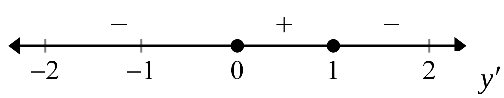
- At a <u>left</u> endpoint *a*: If $f' < 0$ for $x > a$, then f has a local **maximum** value at *a*. If $f' > 0$ for $x > a$, then *f* has a local **minimum** value at *a*.
- At a <u>right</u> endpoint *b*: If $f' < 0$ for $x < b$, then f has a local **maximum** value at *b*. If $f' > 0$ for $x < b$, then f has a local **minimum** value at *b*.

Note: We will use closed intervals (include the endpoints) to describe increasing and decreasing because of the above conditions.

Ex1: Find the critical points of the functions. Find the functions' local and absolute extreme values. Identify the intervals on which f is increasing and decreasing.

1. $f(x) = -2x^3 + 3x^2 - 3$

Critical Points: $f'(x) = -6x^2 + 6x = 0 \Rightarrow -6x(x-1) = 0 \Rightarrow x = 0, 1$



Check for extreme values (sign chart):

When creating a sign chart for increasing/decreasing, be sure to test values in the first derivative! Label the sign chart. When justifying the answer, a sign chart is not sufficient. Explanation must be written out.

Explanation:

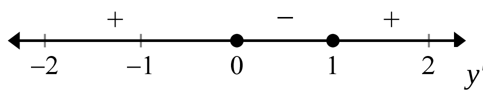
Local Minimum of −3 at $x = 0$, because $f'(x)$ changes sign from – to + at $x = 0$ and $f(0) = -3$; Local Maximum of -2 at $x = 1$, because $f'(x)$ changes sign from + to − at $x = 1$ and $f(1) = -2$;

No absolute extrema | Increasing: $[0, 1]$ | Decreasing: $(-\infty, 0] \cup [1, \infty)$

2. $g(x) = xe^{\frac{1}{x}}$

Critical Points: $g'(x) = e^{\frac{1}{x}} \cdot 1 + x \cdot e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) = e^{\frac{1}{x}} - \frac{e^{\frac{1}{x}}}{x} = \frac{xe^{\frac{1}{x}} - e^{\frac{1}{x}}}{x} = \frac{e^{\frac{1}{x}}(x-1)}{x}$

$g'(x) = 0$: $e^{\frac{1}{x}}(x-1) = 0 \Rightarrow x = 1$ $\qquad$ $g'(x)$ d.n.e.: $x = 0$



Check for extreme values (sign chart):

Local Maximum: none ($g'(x)$ changes sign from + to – at $x = 0$, however $g(0)$ does not exist.)

Local Minimum of *e* at $x = 1$. No absolute extrema

Increasing: $(-\infty, 0) \cup [1, \infty)$ $\qquad$ Decreasing: $(0, 1]$

<u>Concavity</u>: the graph of a differentiable function $y = f(x)$ is

- **Concave Up** on an open interval *I* if y' is increasing on *I*. (SMILES ☺)
  - The tangent lines of a graph that is concave up lie **below** the curve.
- **Concave Down** on an open interval *I* if y' is decreasing on *I*. (FROWNS ☹)
  - The tangent lines of a graph that is concave down lie **above** the curve.

<u>Concavity Test:</u> the graph of a twice-differentiable function $y = f(x)$ is

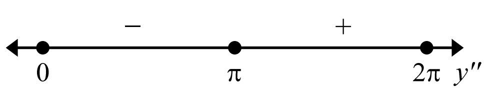
- **Concave Up** on any interval where $y'' > 0$
- **Concave Down** on any interval where $y'' < 0$

<u>Point of Inflection</u>: a point where the graph of a function has a tangent line and where the concavity **changes**

Ex2: Determine the concavity of $y = 2 + \sin x$ on $[0, 2\pi]$ and any point(s) of inflection.

$y' = \cos x$ $\qquad$ $y'' = -\sin x$ $\qquad$ $y'' = 0$ **:** $-\sin x = 0 \Rightarrow x = 0, \pi, 2\pi$ (possible points of inflection)

Make a sign chart:



When creating a sign chart for concavity, be sure to test values in the second derivative! Label the sign chart.

$y = 2 + \sin x$ is concave down on the interval $(0, \pi)$ and concave up on the interval $(\pi, 2\pi)$.

Point of inflection: $\boxed{(\pi, 2)}$; because y'' changes sign at $x = \pi$

<u>Applications of Derivatives:</u>

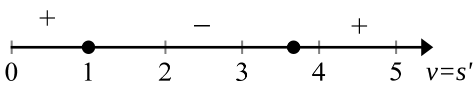
- First Derivative = velocity (sign determines direction)
- Second Derivative = acceleration (sign determines slowing down or speeding up)

Ex3: A particle is moving along a horizontal line with position function $s(t)=2t^3-14t^2+22t-5,\ t\ge 0$. Find the velocity and acceleration functions, and describe the motion of the particle.

$\boxed{v(t)=s'(t)=6t^2-28t+22}$ $\boxed{a(t)=s''(t)=12t-28}$

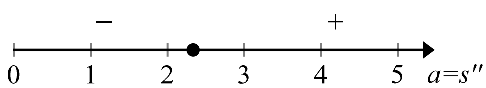
$v(t)=0:\ 6t^2-28t+22=0 \Rightarrow 2(3t^2-14t+11)=0 \Rightarrow 2(3t-11)(t-1)=0 \qquad t=\frac{11}{3},1$

Make a sign chart (velocity):



$a(t)=0:\ 12t-28=0 \Rightarrow t=\frac{7}{3}$

Make a sign chart (acceleration):



Describe the motion: The particle is moving to the right on the interval $(0,1)$, stops at $t=1$ and moves left on the interval $\left(1,\frac{11}{3}\right)$. It then moves right for all $t>\frac{11}{3}$. The particle is decelerating (slowing down) on the interval $\left(0,\frac{7}{3}\right)$ and accelerating (speeding up) for all $t\ge\frac{7}{3}$.

<u>Second Derivative Test for Local Extrema</u>

- If $f'(c)=0$ and $f''(c)<0$, then f has a local **maximum** at $x=c$.
    - f is concave down, so c is at the top of the "hill"
- If $f'(c)=0$ and $f''(c)>0$, then f has a local **minimum** at $x=c$.
    - f is concave up, so c is at the bottom of the "valley"

Ex: Find where the extreme values of $f(x)=x^3-12x-5$ occur using the 2nd Derivative Test.

$f'(x)=3x^2-12$ Critical points: $3x^2-12=0 \Rightarrow x=\pm 2$

$f''(x)=6x$ $f''(-2)=-12<0$ $f''(2)=12>0$

f has a local maximum at $x=-2$ and a local minimum at $x=2$

Curve Sketching

- Identify domain and range, intercepts, asymptotes, and end behavior
- Find critical points and determine increasing/decreasing
- Determine concavity and points of inflection
- Plot a few points if necessary to place the curve

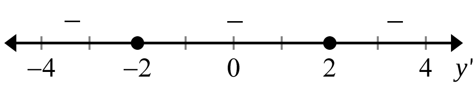
Ex4: Sketch the graph of $f(x)=\dfrac{2x}{x^2-4}$.

Domain: $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$ Range: All Reals *x*-Intercept & *y*-Intercept: $(0,0)$

Vertical Asymptotes: $x=-2,\ x=2$ Horizontal Asymptote: $y=0$

Critical Points: $f'(x)=\dfrac{(x^2-4)2-2x(2x)}{(x^2-4)^2}=\dfrac{-2x^2-8}{(x^2-4)^2}$ $-2x^2-8=0$ No real solution $(x^2-4)^2=0$ $x=-2,\ 2$

Sign Chart:



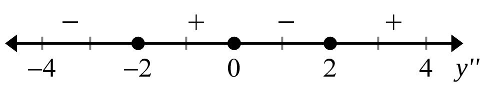
Decreasing: $(-\infty,-2]\cup[-2,2]\cup[2,\infty)$; Never Increasing

Possible points of inflection:

$$f''(x)=\frac{(x^2-4)^2(-4x)-(-2x^2-8)(2(x^2-4)(2x))}{(x^2-4)^4}=\frac{(x^2-4)(-4x^3+16x+8x^3+32x)}{(x^2-4)^4}=\frac{4x^3+48x}{(x^2-4)^3}$$

$4x^3+48x=0 \Rightarrow 4x(x^2+12)=0 \Rightarrow x=0$ $(x^2-4)^3=0 \Rightarrow x=\pm2$

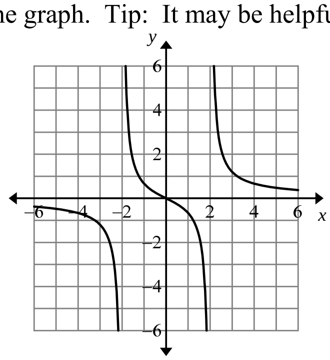
Sign Chart:



Concave Up: $(-2,0)\cup(2,\infty)$ Concave Down: $(-\infty,-2)\cup(0,2)$

Point of Inflection at $x=0$. Note: $x=\pm2$ are not points of inflection because they are not in the domain of *f*.

Sketch the graph. Tip: It may be helpful to make a table or place the sign charts on top of each other.



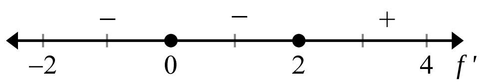
Teacher Note: Have students draw asymptotes with dashed lines.

Curve Sketching Given f'

Ex5: Let $f'(x)=4x^3-8x^2$. Sketch a possible graph for *f*. Write a possible equation for *f*.

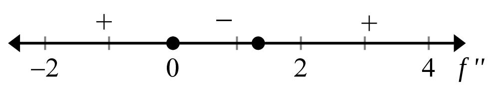
Critical Points: $4x^3-8x^2=0 \Rightarrow 4x^2(x-2)=0 \Rightarrow x=0,2$

Sign Chart:



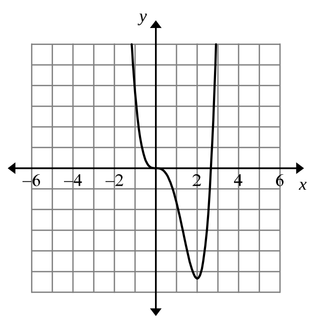
Possible points of inflection: $f''(x)=12x^2-16x=0 \Rightarrow 4x(3x-4)=0 \Rightarrow x=0,\frac{4}{3}$

Sign Chart:

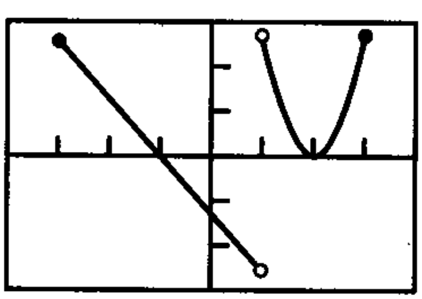


Possible Graph:

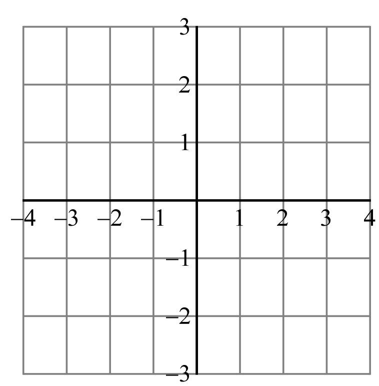
Possible Equation: $f(x)=x^4-\frac{8}{3}x^3$ or $f(x)=x^4-\frac{8}{3}x^3+C$



You Try: Sketch a possible graph of a continuous function *f* that has domain $[-3,3]$, where $f(-1)=1$ and the graph of $y=f'(x)$ is shown below.



[–4, 4] by [–3, 3]

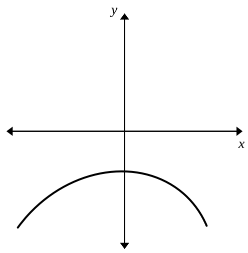


QOD: Explain how the first and second derivatives relate to the graph of a function.

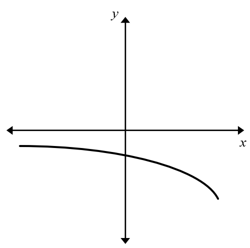
Sample AP Calculus AB Exam Question(s):

1. The function f has the property that $f(x), f'(x),$ and $f''(x)$ are negative for all real values x. Which of the following could be the graph of f?

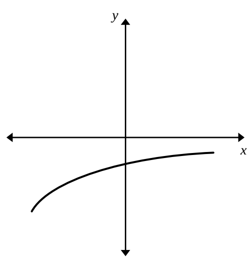
(A)



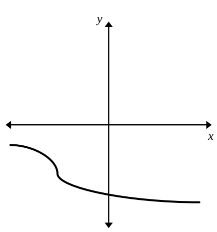
(B)



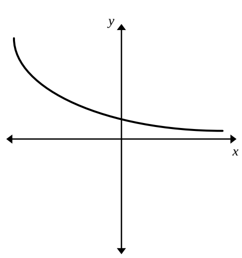
(C)



(D)



(E)



2. Let f be the function given by $f(x) = 2xe^x$. The graph of f is concave down when

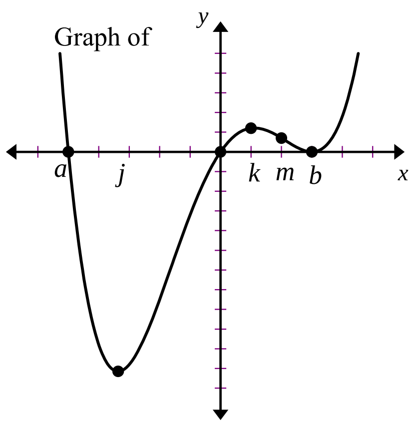
(A) $x < -2$

(B) $x > -2$

(C) $x < -1$

(D) $x > -1$

(E) $x < 0$



3. The second derivative of the function f is given by $f''(x) = x(x-a)(x-b)^2$. The graph of f'' is shown above. For what values of *x* does the graph of f have a point of inflection?

(A) 0 and *a* only

(B) 0 and *m* only

(C) *b* and *j* only

(D) 0, *a*, and *b*

(E) *b*, *j*, and *k*

4. Let *g* be a twice-differentiable function with $g' > 0$ and $g''(x) > 0$ for all real numbers *x*, such that $g(4) = 12$ and $g(5) = 18$. Of the following, which is a possible value for $g(6)$?

(A) 15 (B) 18 (C) 21 (D) 24 (E) 27

5. The function f has first derivative given by $f'(x) = \dfrac{\sqrt{x}}{1 + x + x^3}$. What is the *x*-coordinate of the inflection point of the graph of *f*?

(A) 1.008 (B) 0.473 (C) 0 (D) −0.278 (E) The graph of *f* has no inflection point.

6. For all x in the closed interval $[2,5]$, the function f has a positive first derivative and a negative second derivative. Which of the following could be a table of values for f?

(A)

x	$f(x)$
2	7
3	9
4	12
5	16

(B)

x	$f(x)$
2	7
3	11
4	14
5	16

(C)

x	$f(x)$
2	16
3	12
4	9
5	7

(D)

x	$f(x)$
2	16
3	14
4	11
5	7

(E)

x	$f(x)$
2	16
3	13
4	10
5	7