**Syllabus Objective: 9.1 – The student will sketch the graph of a <u>exponential</u>, logistic, or logarithmic function. 9.2 – The student will evaluate <u>exponential</u> or logarithmic expressions.**

<u>Exponential Function:</u> a function of the form $f(x) = a \cdot b^x$; $a \neq 0, b \neq 1, b > 0$

$a$: initial value ($y$-intercept); $f(0) = ab^0 = a$ $\qquad$ $b$: base

<u>Evaluating an Exponential Function</u>

**Ex:** Determine the value for $f(x) = 3 \cdot 4^x$.

$\bigtriangledown$ Note: A common mistake is multiplying the 3 and 4. Remind students of the order of operations (exponents before multiplication). Or have students write the function as $f(x) = 3(4)^x$.

1. $f(3)$ $\qquad$ $f(3) = 3 \cdot 4^3 = 3 \cdot 64 = \boxed{192}$
2. $f(-2)$ $\qquad$ $f(-2) = 3 \cdot 4^{-2} = 3 \cdot \frac{1}{4^2} = \boxed{\frac{3}{16}}$
3. $f\left(\frac{1}{2}\right)$ $\qquad$ $f\left(\frac{1}{2}\right) = 3 \cdot 4^{\frac{1}{2}} = 3 \cdot \sqrt{4} = 3 \cdot 2 = \boxed{6}$

<u>Writing an Exponential Function Given the Initial Value</u>

**Ex:** Find an exponential function that contains the points $(0,3)$ and $(2,12)$.

Initial Value: $a = 3$ $\qquad$ $y = a \cdot b^x$

Substitute $a = 3$ and $(x, y) = (0,3)$ to find $b$. $\qquad$ $12 = 3 \cdot b^2 \Rightarrow 4 = b^2 \Rightarrow b = \pm 2$

Note: As stated above, $b > 0$, so we will use only $b = 2$. $\qquad$ $\boxed{y = 3 \cdot 2^x}$

<u>Writing an Exponential Function Given Two Points</u>

**Ex:** Find an exponential function that contains the points $(1,12)$ and $(2,36)$.

Substitute each point into $y = a \cdot b^x$ for $x$ and $y$. $\qquad$ $12 = a \cdot b^1$ $\qquad$ $36 = a \cdot b^2$

Divide the two equations to eliminate $a$. $\qquad$ $\frac{36}{12} = \frac{a \cdot b^2}{a \cdot b^1} \Rightarrow 3 = b$

Substitute this value for $b$ to solve for $a$. $\qquad$ $12 = a \cdot 3^1 \Rightarrow 4 = a$

Solution: $\boxed{y = 4 \cdot 3^x}$

<u>Note</u>: Another method for solving the system of equations above is to solve for a and use the substitution method.

Writing an Exponential Function Given a Table of Values

**Ex:** Determine a formula for the table of values.

| $x$ | $-2$ | $-1$ | 0 | 1 |
|---|---|---|---|---|
| $y$ | 63 | 21 | 7 | 7/3 |

$$\frac{21}{63}=\frac{1}{3},\ \frac{7}{21}=\frac{1}{3},\ \frac{7/3}{7}=\frac{1}{3}$$

Check for a common ratio ($b$): $b=\frac{1}{3}$

Initial Value ($a$): value of $y$ when $x=0$: $a=7$

Solution: $\boxed{y=7\left(\frac{1}{3}\right)^x}$

Note: If the table did not give the initial value, then we can use one of the points given for $x$ and $y$, and $b=\frac{1}{3}$ in the equation $y=a\cdot b^x$ to solve for $a$.

**Challenge Problem**: Find $f$ such that $f(m+n)=f(m)f(n)$ for all $m$ and $n$.

Students should recall that when multiplying exponentials with the same base, the exponents are added. So it should make sense to hypothesize that an exponential function would work for $f$.

Try a specific example: $f(x)=2^x$, with $m=3,\ n=2$.

$f(m+n)=f(3+2)=f(5)$ $\qquad$ $f(m)\cdot f(n)=f(3)\cdot f(2)=8\cdot 4=32$

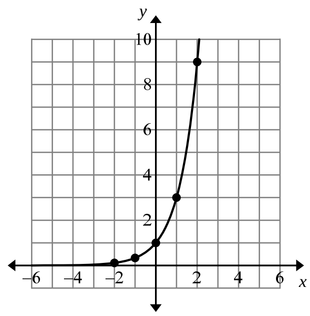
$f(5)=2^5=32$ $\qquad$ $f(3)=2^3=8 \quad f(2)=2^2=4$

Possible Solution: $f(x)=2^x$

Graphs of Exponential Functions

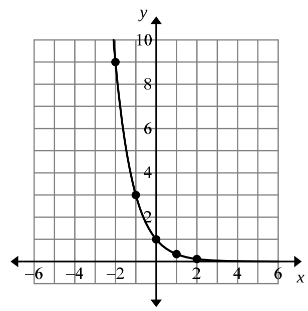
**Ex:** Plot points and graph the functions $y_1 = 3^x$ and $y_2 = \frac{1}{3}^x$.

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y_1$ | 1/9 | 1/3 | 1 | 3 | 9 |
| $y_2$ | 9 | 3 | 1 | 1/3 | 1/9 |




$y_1 = 3^x$

Exponential Growth




$y_2 = \frac{1}{3}^x$

Exponential Decay

Note: Each of the graphs above have a horizontal asymptote at $y = 0$.

Graphs of Exponential Functions: Graph is INCREASING when $b > 1$ & $a > 0$; graph is DECREASING when $0 < b < 1$ & $a > 0$.

Transformations of Exponential Functions: $y = a \cdot b^{(x-h)} + k$

Vertical Stretch: $|a| > 1$ Vertical Shrink: $|a| < 1$

Reflection over $x$-axis: $-a$ Reflection over $y$-axis: $-x$

Horizontal Translation: $h$ Vertical Translation: $k$

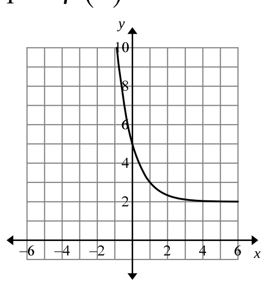
Horizontal Asymptote: $y = k$

**Ex:** Graph the exponential function $y = 3^{(1-x)} + 2$.

Transformation of $y = 3^x$. Rewrite as $y = 3^{-(x-1)} + 2$.

Horizontal Translation: right 1 Reflect over $y$-axis. Vertical Translation: up 2

$y$-intercept: $f(0) = 3^1 + 2 = 5$ Horizontal Asymptote: $y = 2$



Exponential Model: $y = ab^x$

- Exponential Growth: $a > 0$ & $b > 1$; $b$ is the **growth factor**
- Exponential Decay: $a > 0$ & $b < 1$; $b$ is the **decay factor**

Applications of the Exponential Model

**Ex:** CCSD's student population went from 20,420 in 1956 to 291,510 students in 2005. Write an exponential function that represents the student population. Predict the population in 2010.

Let $x = 0$ represent the year 1950 and *y* represent the number of students.

Substitute the given values into the exponential model $y = ab^x$ and solve for *a* and *b*.

$$20420 = a \cdot b^6 \qquad 291510 = a \cdot b^{55}$$

$$\frac{291510}{20420} = \frac{a \cdot b^{55}}{a \cdot b^6} \Rightarrow 14.2757 = b^{49} \Rightarrow 14.2757^{\frac{1}{49}} = b \Rightarrow b \approx 1.056$$ (Exponential Growth)

$$20420 = a \cdot (1.056)^6 \Rightarrow a \approx 14725.57 \qquad \boxed{y = 1475.57(1.056)^x}$$

Population in 2010: $y = 1475.57(1.056)^{60} \approx \boxed{387{,}175 \text{ students}}$

Compound Interest: $A = P\left(1 + \frac{r}{n}\right)^{nt}$ *A* = balance amount *P* = principal (beginning) amount

*r* = annual interest rate (decimal) *n* = # of times compounded in a year *t* = time in years

Note: Annually = 1 time per year Semiannually = 2 times per year

Quarterly = 4 times per year Monthly = 12 times per year Weekly = 52 times per year

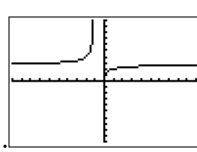
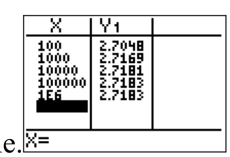
**Ex:** Calculate the balance if $3000 is invested for 10 years at 6% compounded weekly.

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ $A = ?$ $P = 3000$ $t = 10$ $r = 0.06$ $n = 52$

$$A = 3000\left(1 + \frac{0.06}{52}\right)^{52 \cdot 10} \approx \boxed{\$5464.47}$$

Isolating the change in the compounding period reveals a naturally occurring constant. Let the compounding period (*n*) be equal to some constant (*m*) multiplied by the rate: $n = mr$.

$$A = P\left(1 + \frac{r}{mr}\right)^{mrt} = P\left(1 + \frac{1}{m}\right)^{mrt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}$$

Graph $y = \left(1 + \frac{1}{m}\right)^m$ and find $\lim_{m \to \infty}\left(1 + \frac{1}{m}\right)^m$.  Make a table. 

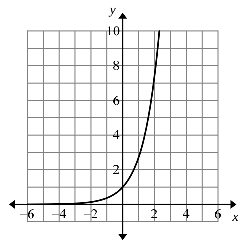

The values of *y* are approaching 2.7183. This is the approximate value of the transcendental number *e*.

<u>Natural Base e</u>: $e=\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x\approx 2.7182818$

<u>Interest Compounded Continuously</u>: $A=Pe^{rt}$

<u>Natural Exponential Function</u>: $f(x)=a\cdot e^{kx}$

Graph of $y=e^x$:



<u>Logistic Function</u>: $f(x)=\frac{c}{1+a\cdot b^x}$ $c$ (constant): limit to growth (maximum)

The logistic function is used for populations that will be limited in their ability to grow due to limited resources or space.

Think About It: What would limit population growth?

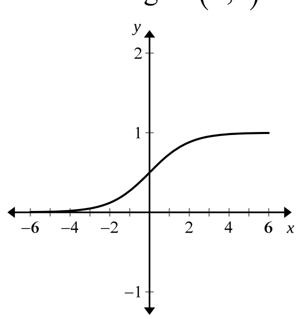
**Ex:** Estimate the maximum population for Dallas and find the population for the year 2008 given the function $P(t)=\frac{1,301,642}{1+21.602e^{-0.05054t}}$ that models the population from 1900.

Maximum population: $c=1,301,642$

Population in 2008: $t=108$ $P(108)=\frac{1,301,642}{1+21.602e^{-0.05054(108)}}\approx 1,191,936$

Graph of a Logistic Function: $f(x)=\frac{1}{1+e^{-x}}$

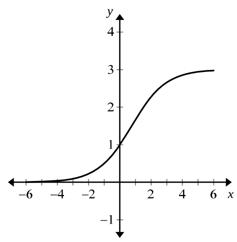
Domain: All Reals Range: $(0,1)$ Always Increasing Horizontal Asymptotes: $y=0, y=1$



**Ex:** Sketch the graph of the function. State the *y*-intercept and horizontal asymptotes.

$$y = \frac{3}{1 + 2 \cdot 0.4^x}$$

*y*-intercept: $y(0) = \frac{3}{1 + 2 \cdot 0.4^0} = \frac{3}{1+2} = 1$  H.A.: $y = 0,\ y = 3$




<u>You Try</u>: Describe the transformations needed to draw the graph of $f(x) = -3 \cdot 2^{x+1} - 4$. Sketch the graph.

<u>QOD</u>: Using a table of values, how can you determine whether they have an exponential relationship?

**Syllabus Objectives: 9.7 – The student will solve application problems involving exponential and logarithmic functions. 9.4 – The student will solve exponential, logarithmic and logistic equations and inequalities.**

Exponential Model: $P(t) = P_0 b^t$ $P(t)$: population at time $t$ $P_0$: initial population

Growth Model: $b = 1 + r$; $b$ is called the **Growth Factor**

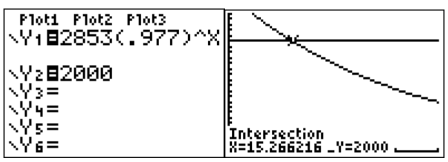
Decay Model: $b = 1 - r$; $b$ is called the **Decay Factor**

**Ex:** Write an exponential function that models the population of Smallville if the initial population was 2,853, and it is decreasing by 2.3% each year. Predict how long it will take for the population to fall to 2000.

$P_0 = 2853$ $r = 0.023$, so $b = 1 - 0.023 = 0.977$

$$\boxed{P(t) = 2853(0.977)^t}$$

Solve for $t$ when $P(t) = 2000$. $2000 = 2853(0.977)^t$




It will take about 15.266 years.

**Ex:** The population of ants is increasing exponentially such that on day 2 there are 100 ants, and on day 4 there are 300 ants. How many ants are there on day 5?

Use $P(t) = P_0 b^t$ and write a system of equations with the given information:

$$100 = P_0 b^2 \qquad 300 = P_0 b^4$$

Solve the system by dividing the equations: $\dfrac{300}{100} = \dfrac{P_0 b^4}{P_0 b^2} \Rightarrow 3 = b^2 \Rightarrow b = \sqrt{3} \approx 1.732$

$$100 = P_0 b^2 \Rightarrow 100 = P_0\left(\sqrt{3}\right)^2 \Rightarrow 100 = 3P_0 \Rightarrow P_0 = \frac{100}{3}$$

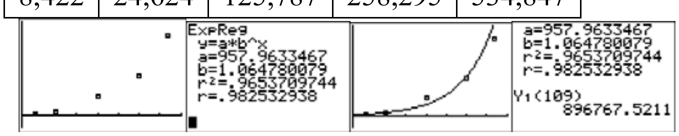
$$P(t) = \frac{100}{3}\left(\sqrt{3}\right)^t \Rightarrow P(5) = \frac{100}{3}\left(\sqrt{3}\right)^5 \approx \boxed{519 \text{ ants}}$$

Note: Students could have called day 2 $t = 0$ to come up with the same solution.

Exponential Regression

**Ex:** Find an exponential regression for the population of Las Vegas using the table below. Then predict the population in 2009.

| 1940 | 1950 | 1970 | 1990 | 2004 |
|---|---|---|---|---|
| 8,422 | 24,624 | 125,787 | 258,295 | 534,847 |




2009: $\boxed{896,767}$

<u>Radioactive Decay</u>: the process in which the number of atoms of a specific element change from a radioactive state to a nonradioactive state

<u>Half-Life</u>: the time it takes for half of a sample of a radioactive substance to change its state

**Ex:** The half-life of radioactive Strontium is 28 days. Write an equation and predict the amount of a 50 gram sample that remains after 100 days.

Use $P(t)=P_0b^t$: Solve for $b$ when $P_0=1$, $P=0.5$, and $t=28$.

$0.5=1b^{28} \Rightarrow b=(0.5)^{\frac{1}{28}} \Rightarrow b\approx 0.9755$ Equation: $P(t)=P_0(0.9755)^t$

50-gram sample remaining after 100 day: $P(t)=50(0.9755)^{100}\approx \boxed{4.185 \text{ grams}}$

<u>Writing a Logistic Function</u>

**Ex:** Find a logistic function that satisfies the given conditions: Initial value = 40; limit to growth = 100; passes through the point $(1,10)$.

$f(x)=\dfrac{c}{1+a\cdot b^x}$ $c=100,\ a=40$ $(x,y)=(1,10)$ $b=?$

$$10=\frac{100}{1+40b^1} \Rightarrow 10+400b=100 \Rightarrow 400b=90 \Rightarrow b=\frac{9}{40}$$

$$\boxed{f(x)=\frac{100}{1+40\cdot\left(\frac{9}{40}\right)^x}}$$

<u>You Try</u>: Complete the table.

| Isotope | Half-Life (years) | Initial Quantity | Amount After 1000 Years |
|---|---|---|---|
| $^{14}C$ | 5730 | 5 grams | |
| $^{14}C$ | 5730 | | 0.7 gram |
| $^{226}Ra$ | 1600 | 100 grams | |

<u>QOD</u>: Explain how to determine if an exponential function is a growth or decay model.

**Syllabus Objective: 9.1 – The student will sketch the graph of an exponential, logistic or <u>logarithmic</u> function. 9.2 – The student will evaluate exponential or <u>logarithmic</u> expressions. 9.5 – The student will graph the inverse of an exponential or logarithmic function.**

<u>Review</u>: Solve the following for *x*.

1. $10^x = 1000$ $\qquad\qquad$ $x = 3$
2. $10^x = 0.01$ $\qquad$ $10^x = \dfrac{1}{100} = \dfrac{1}{10^2}$ $\qquad$ $x = -2$
3. $10^x = 100\sqrt{10}$ $\qquad$ $10^x = 10^2\left(10^{\frac{1}{2}}\right) = 10^{2+\frac{1}{2}}$ $\qquad$ $x = \dfrac{5}{2}$
4. $10^x = \dfrac{1}{\sqrt[3]{100}}$ $\qquad$ $10^x = \dfrac{1}{\sqrt[3]{10^2}} = \dfrac{1}{10^{\frac{2}{3}}}$ $\qquad$ $x = -\dfrac{2}{3}$

<u>Exploration</u>: Use your calculator to write 1997 as a power of 10.

Note: $1000 = 10^3$, $10000 = 10^4$

Try $10^{3.1} \approx 1258.9,\ 10^{3.2} \approx 1584.9,\ 10^{3.3} \approx 1995.3$ (close!)

Now use your calculator to find $\text{LOG}(1997)$. $\log(1997) \approx 3.300378$ (Compare to above)

<u>Common Logarithm</u>: Given a positive number *p*, the solution to $10^x = p$ is called the base-10 logarithm of p, expressed as $x = \log_{10} p$, or simply as $x = \log p$. (When no base is specified, it is understood to be base 10.)

A LOGARITHM IS AN **EXPONENT**.

<u>Logarithm (base *b*):</u> $y = \log_b x$ (Read as "log base *b* of *x*.") for $x > 0,\ b > 0$ and $b \neq 1$ if and only if $x = b^y$.

<u>Note</u>: You cannot take the log of a negative number!

**Ex:** Rewrite each equation (exponential form) to logarithmic form.

1. $10^3 = 1000$ $\qquad$ Base (*b*) = 10 $\qquad$ Exponent = 3 $\qquad$ $\log_{10} 1000 = 3$ or $\boxed{\log 1000 = 3}$
2. $2^4 = 16$ $\qquad$ Base (*b*) = 2 $\qquad$ Exponent = 4 $\qquad$ $\boxed{\log_2 16 = 4}$

**Ex:** Rewrite each equation (logarithmic form) to exponential form.

1. $\log_5 125 = 3$ $\qquad$ Base (*b*) = 5 $\qquad$ Exponent = 3 $\qquad$ $\boxed{5^3 = 125}$
2. $\log 100{,}000 = 5$ $\qquad$ Base (*b*) = 10 $\qquad$ Exponent = 5 $\qquad$ $\boxed{10^5 = 100{,}000}$

Evaluating Logarithms: a logarithm is an **exponent**

**Ex:** Evaluate the logarithms.

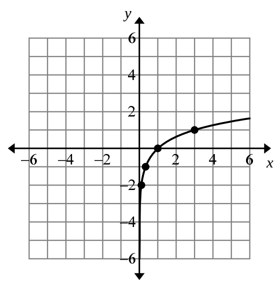
1. $y = \log_3 81$ $\qquad 3^y = 81$ $\qquad \boxed{y = 4}$
2. $f\left(\frac{1}{16}\right),\ f(x) = \log_4 x$ $\qquad f\left(\frac{1}{16}\right) = \log_4\left(\frac{1}{16}\right)$ $\qquad 4^y = \frac{1}{16}$ $\qquad \boxed{y = -2}$
3. $y = \log_5 1$ $\qquad 5^y = 1$ $\qquad \boxed{y = 0}$
4. $f(2.2),\ f(x) = \log x$ $\qquad f(x) = \log 2.2$ $\qquad$ Calculator: $\boxed{f(x) \approx 0.342}$

Graphing Logarithmic Functions: a logarithmic function is the inverse of an exponential function

**Ex:** Use a table of values to sketch the graph of $y = \log_3 x$. Discuss the characteristics of the graph and compare the graph to the graph of $y = 3^x$.

Note: To create the table, it is helpful to rewrite the function as $3^y = x$ and choose values for *y*.

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 9 | 3 | 1 | 1/3 | 1/9 |
| $y$ | 2 | 1 | 0 | −1 | −2 |




Domain: $(0,\infty)$ $\qquad$ Range: $(-\infty,\infty)$ $\qquad$ Intercepts: $x-\text{int}: (1,0)$ $\qquad$ Asymptote: $x = 0$

Increasing: $(0,\infty)$ $\qquad$ End Behavior: $\lim_{x\to 0} f(x) = -\infty$ $\qquad \lim_{x\to\infty} f(x) = \infty$

The function $y = \log_3 x$ is the inverse of $y = 3^x$, so its graph is the reflection of $y = 3^x$ over the line $y = x$.

Transformations of $y = \log_b x$: $\qquad y = k + a \cdot \log_b (x - h)$

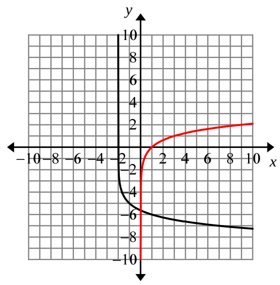
Vertical Stretch: $a,\ |a| > 1$ $\qquad$ Horizontal Stretch: $a,\ |a| < 1$

Reflection over *x*-axis: $-a$ $\qquad$ Reflection over *y*-axis: $-x$

Horizontal Translation: $h$ $\qquad$ Vertical Translation: $k$

**Ex:** Describe the transformations used to graph the function $y = -\log_3(x+2) - 5$. Then sketch the graph.

Reflect over *x*-axis; shift left 2 units; shift down 5 units




Note: Graph in red is the graph of $y = \log_3 x$.

Properties of Logarithms:

- $\log_a 1 = 0$ because $a^0 = 1$
- $\log_a a = 1$ because $a^1 = a$
- $\log_a a^x = x$ because $a^x = a^x$ or $a^{\log_a x} = x$ (property of inverses)
- If $\log_a x = \log_a y$, then $x = y$

Natural Logarithmic Function: logarithm base *e*, $\log_e x$, written $y = \ln x$

Notations:

1. $\log x$ is used to represent $\log_{10} x$ (common logarithm)
2. $\ln x$ is used to represent $\log_e x$ (natural logarithm)

Evaluating Logarithmic Expressions:

**Ex:** Evaluate $\log\left(10^{-2.5}\right)$ and $10^{\log 4.1}$.

$\log\left(10^{-2.5}\right)$ Let $y = \log\left(10^{-2.5}\right)$. Rewrite in exponential form: $10^y = 10^{-2.5}$ $\boxed{y = -2.5}$

$10^{\log 4.1}$ Let $y = 10^{\log 4.1}$. Rewrite in logarithmic form: $\log y = \log 4.1$ $\boxed{y = 4.1}$

Note: By the properties of inverses, we could have evaluated the above examples without rewriting using the following: $\log_a a^x = x$ and $a^{\log_a x} = x$

Solving Logarithmic Equations:

**Ex:** Solve the equations for *x*.

1. $\log_7 x^2 = \log_7 36$ The bases are equal, so $x^2 = 36 \Rightarrow \boxed{x = \pm 6}$.
   Note: Both solutions work in the original equation.
2. $\log_9 3 = x$ Rewrite exponentially: $9^x = 3 \Rightarrow \boxed{x = \frac{1}{2}}$
3. $\ln e^4 = x$ Rewrite exponentially: $e^x = e^4 \Rightarrow \boxed{x = 4}$
   Note: Remember that the base of ln is *e*.

You Try: Sketch the graph of function on the same grid with its inverse. $f(x) = \ln x$

QOD: Can you evaluate the log of a negative number? Explain.

**Syllabus Objective: 9.3 – The student will apply the properties of logarithms to evaluate expressions, change bases, and re-express data.**

Exploration: Use your calculator to find $\log 2$ and $\log 3$. Evaluate the following logarithms on your calculator, then speculate how you could calculate them using only the values of $\log 2$ and/or $\log 3$.

$\log 2 \approx 0.301$ and $\log 3 \approx 0.477$

1. $\log 6$ $\quad \log 6 \approx 0.778 \quad 0.778 = 0.301 + 0.477 = \log 2 + \log 3$
2. $\log 8$ $\quad \log 8 \approx 0.903 \quad 0.903 = 3 \cdot 0.301 = 3\log 2$
3. $\log \frac{2}{3}$ $\quad \log \frac{2}{3} = -0.176 \quad -0.176 = 0.301 - 0.477 = \log 2 - \log 3$

Recall: Properties of Exponents $\quad b^x \cdot b^y = b^{x+y} \quad \frac{b^x}{b^y} = b^{x-y} \quad \left(b^x\right)^y = b^{x \cdot y}$

Because a logarithm is an exponent, the rules are the same!

Properties of Logarithms:

$$\log_b(RS) = \log_b R + \log_b S$$

$$\log_b\left(\frac{R}{S}\right) = \log_b R - \log_b S$$

$$\log_b\left(R^c\right) = c \cdot \log_b R$$

**Ex:** Use the properties of logarithms to expand the following expressions.

1. $\log(5x)$ $\quad \log(5x) = \boxed{\log 5 + \log x}$
2. $\ln\left(\frac{xy}{z}\right)$ $\quad \ln\left(\frac{xy}{z}\right) = \boxed{\ln x + \ln y - \ln z}$
3. $\log_2\left(\frac{x}{\sqrt{x^2+1}}\right)$ $\quad \log_2\left(\frac{x}{\sqrt{x^2+1}}\right) = \log_2 x - \log\left(x^2+1\right)^{\frac{1}{2}} = \boxed{\log_2 x - \frac{1}{2}\log\left(x^2+1\right)}$
4. $\log\left(200x^2\right)$

$$\log 200 + \log x^2 = \log(100 \cdot 2) + 2\log x = \log 100 + \log 2 + 2\log x = \boxed{2 + \log 2 + 2\log x}$$

**Ex:** Use the properties of logarithms to condense the following expressions.

1. $\log_3 x + \log_3 4$ $\qquad \log_3 x + \log_3 4 = \boxed{\log_3(4x)}$
2. $\log x - 3\log(x+1)$ $\qquad \log x - 3\log(x+1) = \log x - \log(x+1)^3 = \boxed{\log \dfrac{x}{(x+1)^3}}$
3. $2\ln x - \ln(x+1) - \ln(x-1)$

$$2\ln x - \ln(x+1) - \ln(x-1) = \ln x^2 - \left(\ln(x+1) + \ln(x-1)\right) = \boxed{\ln \frac{x^2}{(x+1)(x-1)}} = \boxed{\ln \frac{x^2}{x^2-1}}$$

<u>Evaluating Logarithmic Expressions with Base *b*</u>

Let $y = \log_b x$. Rewrite in exponential form: $b^y = x$.

Take the log of both sides. $\qquad \log b^y = \log x$

Use the properties of logs to solve for *y*. $\qquad y\log b = \log x \Rightarrow y = \dfrac{\log x}{\log b}$

Note: This will work for a logarithm of any base, including the natural log.

<u>Change of Base Formula</u>: $\qquad \log_b a = \dfrac{\log a}{\log b} = \dfrac{\ln a}{\ln b}$

**Ex:** Evaluate the expression $\log_5 8$.

By the change of base formula, $\log_5 8 = \dfrac{\log 8}{\log 5}$. Using the calculator, $\dfrac{\log 8}{\log 5} \approx \boxed{1.292}$

**Ex:** Evaluate the expression $\log_{64} 2$.

By the change of base formula, $\log_{64} 2 = \dfrac{\ln 2}{\ln 64}$. Using the calculator, $\dfrac{\ln 2}{\ln 64} \approx \boxed{0.167}$

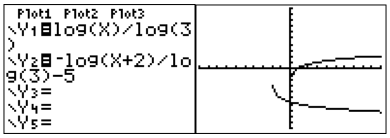
Note – This did NOT require the use of a calculator! We know that $64^{\frac{1}{6}} = 2$. So $\log_{64} 2 = \boxed{\dfrac{1}{6}}$

Graphing Logarithmic Functions on the Calculator

**Ex:** Graph $y = \log_3 x$ and $y = -\log_3(x+2) - 5$ on the same grid on the graphing calculator.

We cannot type in a log base 3 into the calculator, so we must rewrite the functions using the change of base formula.

$y_1 = \log_3 x \Rightarrow y_1 = \dfrac{\log x}{\log 3}$ $\qquad$ $y_2 = -\log_3(x+2) - 5 \Rightarrow y_2 = -\dfrac{\log(x+2)}{\log 3} - 5$




Caution: The graph created by the calculator is misleading at the asymptote!

You Try: Expand the expression using the properties of logarithms. $\ln\left(\dfrac{x^3}{yz}\right)$.

QOD: When is it appropriate to use the change of base formula? Explain how to evaluate a logarithm of base *b* without the change of base formula.

**Syllabus Objectives: 9.4 – The student will solve exponential, logarithmic and logistic equations and inequalities. 9.6 – The student will compare equivalent logarithmic and exponential equations.**

Strategies for Solving an Exponential Equation:

- Rewrite both sides with the same base
- Take the log of both sides after isolating the exponential

**Ex:** Solve the following exponential equations.

1. $4^x = 32$ Rewrite with base 2: $\left(2^2\right)^x = 2^5 \Rightarrow 2^{2x} = 2^5$

   Both sides have the same base, so the exponents must be equal: $2x = 5 \Rightarrow \boxed{x = \frac{5}{2}}$

2. $8^{3x} = 360$ We cannot rewrite both sides with the same base, so take the log of both sides.

   $\log 8^{3x} = \log 360 \Rightarrow 3x \cdot \log 8 = \log 360$ Solve for x: $x = \frac{\log 360}{3\log 8} \Rightarrow \boxed{x \approx 0.944}$

3. $7 - 2e^x = 5$ Isolate the exponential: $-2e^x = -2 \Rightarrow e^x = 1$

   Take the natural log of both sides: $\ln e^x = \ln 1 \Rightarrow x\ln e = \ln 1 \Rightarrow x = \frac{\ln 1}{\ln e} \Rightarrow \boxed{x = 0}$

   Note: We could have determined that $x = 0$ immediately using the equation $e^x = 1$.

Strategies for Solving a Logarithmic Equation:

- Condense any logarithms with the same base using the properties of logs
- Rewrite the equation in exponential form
- Check for extraneous solutions

**Ex:** Solve the following logarithmic equations.

1. $\ln x - \ln 5 = 0$ Condense: $\ln\left(\frac{x}{5}\right) = 0$

   Rewrite in exponential form: $e^0 = \frac{x}{5}$ Solve for *x*: $1 = \frac{x}{5} \Rightarrow \boxed{x = 5}$

   Check: $\ln 5 - \ln 5 = 0$ ☺

2. $\log_x 625 = 4$ Rewrite in exp. form: $x^4 = 625$

   Take the log of both sides: $\ln x^4 = \ln 625 \Rightarrow 4\ln x = \ln 625 \Rightarrow \ln x = \frac{\ln 625}{4}$

   Rewrite in exponential form: $x = e^{\frac{\ln 625}{4}} \Rightarrow \boxed{x = 5}$

   Note: We could have determined that $x = 5$ immediately using the equation $x^4 = 625$, by taking the 4th root of each side.

3. $\log_3 x + \log_3(x-8) = 2$  Condense: $\log_3\left(x(x-8)\right) = 2$

Rewrite in exponential form: $3^2 = x(x-8)$

Solve for *x*: $9 = x^2 - 8x \Rightarrow x^2 - 8x - 9 = 0 \Rightarrow (x-9)(x+1) = 0 \quad x = -1, 9$

Check: $x = -1$: $\log_3(-1) + \log_3(-1-8) \neq 2$ (can't take the log of a negative!)

$x = 9$: $\log_3 9 + \log_3(9-8) = 2 + \log_3 1 = 2 + 0 = 2$ ☺  $\boxed{x = 9}$

Note: You must check every possible solution for extraneous solutions. All negative answers are not necessarily extraneous!

**Ex:** Solve the equation $\log_5(4-x) = 1$.

Rewrite in exponential form: $5^1 = 4 - x$  Solve for *x*: $x = -1$

Check: $\log_5(4-(-1)) = \log_5 5 = 1$ ☺  $\boxed{x = -1}$

**Ex:** Solve the equation $\log x^2 = 6$.

Rewrite using properties of logs: $2\log x = 6 \Rightarrow \log x = 3$

Rewrite in exponential form: $10^3 = x \Rightarrow \boxed{x = 1000}$

Check: $\log(1000^2) = \log\left((10^3)^2\right) = \log(10^6) = 6$ ☺

Challenge Problems: Use your "arsenal" of exponential and logarithmic properties!

**Ex:** Solve the equation $2^{2x} - 6(2)^x - 7 = 0$.

Rewrite the first term: $(2^x)^2 - 6(2^x) - 7 = 0$  Let $2^x = y \Rightarrow y^2 - 6y - 7 = 0$.

Solve for *y*: $(y-7)(y+1) = 0 \Rightarrow y = -1, 7$  Use *y* to solve for *x*: $2^x = y$

$y = -1$: $2^x = -1$ (no solution)

$y = 7$: $2^x = 7 \Rightarrow \log 2^x = \log 7 \Rightarrow x\log 2 = \log 7 \Rightarrow x = \dfrac{\log 7}{\log 2}$  $\boxed{x \approx 2.807}$

**Ex:** Solve the equation $4^{3x} = 7^{(x-1)}$.

Take the log of both sides: $\ln 4^{3x} = \ln 7^{(x-1)}$  Rewrite: $3x \cdot \ln 4 = (x-1)\ln 7$

Solve for *x*: $3x \cdot \ln 4 = x\ln 7 - \ln 7 \Rightarrow (3\ln 4)x - (\ln 7)x = -\ln 7 \Rightarrow x(3\ln 4 - \ln 7) = -\ln 7$

$x = \dfrac{-\ln 7}{3\ln 4 - \ln 7} \Rightarrow \boxed{x \approx -0.879}$  Check ☺

You Try: Solve the equation $\frac{200}{1+e^{-x}}=150$.

QOD: Compare and contrast the methods for solving exponential and logarithmic equations.

**Syllabus Objective: 9.7 – The student will solve application problems involving exponential and logarithmic functions.**

Newton's Law of Cooling: The temperature $T$ of an object at time $t$ is $T(t) = T_s + (T_0 - T_S)e^{-kt}$, where $T_s$ is the surrounding temperature and $T_0$ is the initial temperature of the object.

**Ex:** A 350°F potato is left out in a 70°F room for 12 minutes, and its temperature dropped to 250°F. How many more minutes will it take to reach 120°F?

Solve for *k* using the given information: $T(t) = T_s + (T_0 - T_S)e^{-kt}$

$$250 = 70 + (350 - 70)e^{-12k} \Rightarrow 180 = 280e^{-12k} \Rightarrow \frac{9}{14} = e^{-12k}$$

$$\ln\left(\frac{9}{14}\right) = \ln\left(e^{-12k}\right) \Rightarrow \ln\frac{9}{14} = -12k \Rightarrow k = \frac{\ln\frac{9}{14}}{-12} \approx 0.0368$$

Use *k* to solve for *t*:

$$120 = 70 + (350 - 70)e^{-0.0368t} \Rightarrow \frac{5}{28} = e^{-0.0368t} \Rightarrow \ln\frac{5}{28} = -0.0368t \Rightarrow t \approx 46.8$$

It takes about 46.8 minutes for the potato to cool to 120°F. This is 46.8 – 12 = 35.8 minutes longer.

Formula for Compound Interest: $A = P\left(1 + \frac{r}{n}\right)^{nt}$ $A$ = balance $r$ = annual interest rate

$P$ = principal $t$ = time in years $n$ = number of times interest is compounded each year

Interest Compounded Continuously: $A = Pe^{rt}$

**Ex:** How long will it take for an investment of $2,000 at 6% compounded semi-annually to reach $5000?

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$5000 = 2000\left(1 + \frac{0.06}{2}\right)^{2t} \Rightarrow \frac{5}{2} = 1.03^{2t} \Rightarrow \ln\frac{5}{2} = \ln 1.03^{2t} \Rightarrow \ln\frac{5}{2} = 2t \cdot \ln 1.03$$

$$t = \frac{\ln\frac{5}{2}}{2\ln 1.03} \approx 15.499$$

It will take about 15.5 years.

**Ex:** How long will it take for an investment of \$2,000 at 6% compounded continuously to reach \$5000?

$$A = Pe^{rt} \qquad 5000 = 2000e^{0.06t} \Rightarrow \ln\frac{5}{2} = \ln e^{0.06t} \Rightarrow 0.06t = \ln\frac{5}{2} \Rightarrow t = \frac{\ln\frac{5}{2}}{0.06} \approx 15.27$$

It will take about 15.27 years.

Annual Percentage Yield (APY): the rate, compounded annually ($t = 1$), that would yield the same return

For $A = P\left(1+\frac{r}{n}\right)^{nt}$, $\text{APY} = \left(1+\frac{r}{n}\right)^{n} - 1$

**Ex:** An amount of \$2400 is invested for 8 years at 5% compounded quarterly. What is the equivalent APY?

$$\text{APY} = \left(1+\frac{r}{n}\right)^{n} - 1 = \left(1+\frac{0.05}{4}\right)^{4} - 1 \approx 0.0509 \approx \boxed{5.09\%}$$

You Try: Determine the amount of money that should be invested at 9% interest compounded monthly to produce a balance of \$30,000 in 15 years.

QOD: Why is using the annual percentage yield a more "fair" way to compare investments?