Precalculus Notes: Unit 3 – Exponential and Logarithmic Functions

Syllabus Objective: 9.1 – The student will sketch the graph of an exponential, logistic, or logarithmic function. 9.2 – The student will evaluate exponential or logarithmic expressions.

Exponential Function: a function of the form \( f(x) = a \cdot b^x \); \( a \neq 0, b \neq 1, b > 0 \)

- \( a \): initial value (y-intercept); \( f(0) = a b^0 = a \)
- \( b \): base

Evaluating an Exponential Function

**Ex:** Determine the value for \( f(x) = 3 \cdot 4^x \).

\( f(3) = 3 \cdot 4^3 = 3 \cdot 64 = 192 \)

\( f(-2) = 3 \cdot 4^{-2} = 3 \cdot \frac{1}{4^2} = \frac{3}{16} \)

\( f\left(\frac{1}{2}\right) = 3 \cdot 4^{\frac{1}{2}} = 3 \cdot \sqrt{4} = 3 \cdot 2 = 6 \)

Writing an Exponential Function Given the Initial Value

**Ex:** Find an exponential function that contains the points \((0, 3)\) and \((2,12)\).

Initial Value: \( a = 3 \) \( y = a \cdot b^x \)

Substitute \( a = 3 \) and \((x, y) = (0, 3)\) to find \( b \).

\[ 12 = 3 \cdot b^2 \quad \Rightarrow \quad 4 = b^2 \quad \Rightarrow \quad b = \pm 2 \]

Note: As stated above, \( b > 0 \), so we will use only \( b = 2 \). \[ y = 3 \cdot 2^x \]

Writing an Exponential Function Given Two Points

**Ex:** Find an exponential function that contains the points \((1,12)\) and \((2,36)\).

Substitute each point into \( y = a \cdot b^x \) for \( x \) and \( y \).

\[ 12 = a \cdot b^1 \quad \Rightarrow \quad 36 = a \cdot b^2 \]

Divide the two equations to eliminate \( a \).

\[ \frac{36}{12} = \frac{a \cdot b^2}{a \cdot b^1} \quad \Rightarrow \quad 3 = b \]

Substitute this value for \( b \) to solve for \( a \).

\[ 12 = a \cdot 3^1 \quad \Rightarrow \quad 4 = a \]

Solution: \[ y = 4 \cdot 3^x \]

Note: Another method for solving the system of equations above is to solve for \( a \) and use the substitution method.
Writing an Exponential Function Given a Table of Values

**Ex:** Determine a formula for the table of values.

<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>63</td>
<td>21</td>
<td>7</td>
<td>7/3</td>
</tr>
</tbody>
</table>

Check for a common ratio \( b \): \( b = \frac{1}{3} \)

Initial Value \((a)\): value of \( y \) when \( x = 0 \): \( a = 7 \)

**Solution:** \( y = 7 \left( \frac{1}{3} \right)^x \)

**Note:** If the table did not give the initial value, then we can use one of the points given for \( x \) and \( y \), and \( b = \frac{1}{3} \) in the equation \( y = a \cdot b^x \) to solve for \( a \).

**Challenge Problem:** Find \( f \) such that \( f(m + n) = f(m) \cdot f(n) \) for all \( m \) and \( n \).

Students should recall that when multiplying exponentials with the same base, the exponents are added. So it should make sense to hypothesize that an exponential function would work for \( f \).

Try a specific example: \( f(x) = 2^x \), with \( m = 3 \), \( n = 2 \).

\[
\begin{align*}
    f(m+n) &= f(3+2) = f(5) \\
    f(m) \cdot f(n) &= f(3) \cdot f(2) = 8 \cdot 4 = 32 \\
    f(5) &= 2^5 = 32 \\
    f(3) &= 2^3 = 8 \\
    f(2) &= 2^2 = 4
\end{align*}
\]

Possible Solution: \( f(x) = 2^x \)
Graphs of Exponential Functions

**Ex:** Plot points and graph the functions \( y_1 = 3^x \) and \( y_2 = \frac{1}{3^x} \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-2)</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_1 )</td>
<td>(\frac{1}{9})</td>
<td>(\frac{1}{3})</td>
<td>1</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>( y_2 )</td>
<td>9</td>
<td>3</td>
<td>1/3</td>
<td>1/9</td>
<td></td>
</tr>
</tbody>
</table>

Exponential Growth  
Exponential Decay

Note: Each of the graphs above have a horizontal asymptote at \( y = 0 \).

Graphs of Exponential Functions: Graph is INCREASING when \( b > 1 \) & \( a > 0 \); graph is DECREASING when \( 0 < b < 1 \) & \( a > 0 \).

Transformations of Exponential Functions: \( y = a \cdot b^{(x-h)} + k \)

Vertical Stretch: \(|a| > 1\)  
Vertical Shrink: \(|a| < 1\)

Reflection over \( x \)-axis: \( -a \)  
Reflection over \( y \)-axis: \( -x \)

Horizontal Translation: \( h \)  
Vertical Translation: \( k \)

Horizontal Asymptote: \( y = k \)

**Ex:** Graph the exponential function \( y = 3^{(1-x)} + 2 \).

Transformation of \( y = 3^x \). Rewrite as \( y = 3^{-x+1} + 2 \).

Horizontal Translation: right 1  
Reflect over \( y \)-axis.  
Vertical Translation: up 2

\( y \)-intercept: \( f(0) = 3^1 + 2 = 5 \)  
Horizontal Asymptote: \( y = 2 \)
Exponential Model:  \( y = ab^x \)

- Exponential Growth: \( a > 0 \) & \( b > 1 \); \( b \) is the **growth factor**
- Exponential Decay: \( a > 0 \) & \( b < 1 \); \( b \) is the **decay factor**

**Applications of the Exponential Model**

**Ex:** CCSD’s student population went from 20,420 in 1956 to 291,510 students in 2005. Write an exponential function that represents the student population. Predict the population in 2010.

Let \( x = 0 \) represent the year 1950 and \( y \) represent the number of students.

Substitute the given values into the exponential model \( y = ab^x \) and solve for \( a \) and \( b \).

\[
\frac{291510}{20420} = a \cdot b^{55} \quad \text{and} \quad \frac{1}{b} = 1.056
\]

\[
20420 = a \cdot (1.056)^6 \Rightarrow a \approx 14725.7
\]

\[
y = 1475.57 \cdot 1.056 \Rightarrow y = 1475.57 \cdot 1.056 \approx 387,175 \text{ students}
\]

**Compound Interest:**

\[
A = P \left(1 + \frac{r}{n}\right)^{nt}
\]

\( A \) = balance amount  \( P \) = principal (beginning) amount

\( r \) = annual interest rate (decimal)  \( n \) = # of times compounded in a year  \( t \) = time in years

**Note:**
- Annually = 1 time per year
- Semiannually = 2 times per year
- Quarterly = 4 times per year
- Monthly = 12 times per year
- Weekly = 52 times per year

**Ex:** Calculate the balance if \$3000 is invested for 10 years at 6\% compounded weekly.

\[
\begin{align*}
A &= P \left(1 + \frac{r}{n}\right)^{nt} \\
&= 3000 \left(1 + \frac{0.06}{52}\right)^{52 \cdot 10} \\
&\approx \$5464.47
\end{align*}
\]

Isolating the change in the compounding period reveals a naturally occurring constant. Let the compounding period \( (n) \) be equal to some constant \( (m) \) multiplied by the rate: \( n = mr \).

\[
A = P \left(1 + \frac{r}{mr}\right)^{mrt} = P \left(1 + \frac{1}{m}\right)^{mr} = P \left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}
\]

Graph \( y = \left(1 + \frac{1}{m}\right)^{m} \) and find \( \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{m} \). Make a table.

The values of \( y \) are approaching 2.7183. This is the approximate value of the transcendental number \( e \).
Natural Base $e$:  
$$e = \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x \approx 2.7182818$$

Interest Compounded Continuously:  
$$A = Pe^{rt}$$

Natural Exponential Function:  
$$f(x) = a \cdot e^{kx}$$

Graph of $y = e^x$:

Logistic Function:  
$$f(x) = \frac{c}{1 + a \cdot e^{bx}}$$

$c$ (constant): limit to growth (maximum)

The logistic function is used for populations that will be limited in their ability to grow due to limited resources or space.

Think About It: What would limit population growth?

Example: Estimate the maximum population for Dallas and find the population for the year 2008 given the function  
$$P(t) = \frac{1,301,642}{1 + 21.602e^{-0.05054t}}$$

that models the population from 1900.

Maximum population:  
$$c = 1,301,642$$

Population in 2008:  
$$t = 108 \quad P(108) = \frac{1,301,642}{1 + 21.602e^{-0.05054(108)}} \approx 1,191,936$$

Graph of a Logistic Function:  
$$f(x) = \frac{1}{1 + e^{-x}}$$

Domain: All Reals  
Range: $(0,1)$  
Always Increasing  
Horizontal Asymptotes: $y = 0, y = 1$
Ex: Sketch the graph of the function. State the $y$-intercept and horizontal asymptotes.

$$y = \frac{3}{1 + 2 \cdot 0.4^x}$$

$y$-intercept: $y(0) = \frac{3}{1 + 2 \cdot 0.4^0} = \frac{3}{1 + 2} = 1$

H.A.: $y = 0, y = 3$

You Try: Describe the transformations needed to draw the graph of $f(x) = -3 \cdot 2^{x+1} - 4$. Sketch the graph.

QOD: Using a table of values, how can you determine whether they have an exponential relationship?
Syllabus Objectives: 9.7 – The student will solve application problems involving exponential and logarithmic functions. 9.4 – The student will solve exponential, logarithmic and logistic equations and inequalities.

Exponential Model: \( P(t) = P_0 b^t \)  
- \( P(t) \): population at time \( t \)  
- \( P_0 \): initial population

Growth Model: \( b = 1 + r \); \( b \) is called the Growth Factor

Decay Model: \( b = 1 - r \); \( b \) is called the Decay Factor

Ex: Write an exponential function that models the population of Smallville if the initial population was 2,853, and it is decreasing by 2.3% each year. Predict how long it will take for the population to fall to 2000.

\[ P_0 = 2853 \quad r = 0.023 \, \text{, so } b = 1 - 0.023 = 0.977 \]

\[ P(t) = 2853(0.977)^t \]

Solve for \( t \) when \( P(t) = 2000 \).

\[ 2000 = 2853(0.977)^t \]

It will take about 15.266 years.

Ex: The population of ants is increasing exponentially such that on day 2 there are 100 ants, and on day 4 there are 300 ants. How many ants are there on day 5?

Use \( P(t) = P_0 b^t \) and write a system of equations with the given information:

\[ \begin{align*}
100 &= P_0 b^2 \\
300 &= P_0 b^4
\end{align*} \]

Solve the system by dividing the equations:

\[ \frac{300}{100} = \frac{P_0 b^4}{P_0 b^2} \quad \Rightarrow \quad 3 = b^2 \quad \Rightarrow \quad b = \sqrt{3} \approx 1.732 \]

\[ 100 = P_0 b^2 \quad \Rightarrow \quad 100 = P_0 (\sqrt{3})^2 \quad \Rightarrow \quad 100 = 3 P_0 \quad \Rightarrow \quad P_0 = \frac{100}{3} \]

\[ P(t) = \frac{100}{3} (\sqrt{3})^t \quad \Rightarrow \quad P(5) = \frac{100}{3} (\sqrt{3})^5 \approx 519 \text{ ants} \]

Note: Students could have called day 2 \( t = 0 \) to come up with the same solution.

Exponential Regression

Ex: Find an exponential regression for the population of Las Vegas using the table below. Then predict the population in 2009.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1940</td>
<td>8,422</td>
</tr>
<tr>
<td>1950</td>
<td>24,624</td>
</tr>
<tr>
<td>1970</td>
<td>125,787</td>
</tr>
<tr>
<td>1990</td>
<td>258,295</td>
</tr>
<tr>
<td>2004</td>
<td>534,847</td>
</tr>
</tbody>
</table>

2009: 896,767
Radioactive Decay: the process in which the number of atoms of a specific element change from a radioactive state to a nonradioactive state

Half-Life: the time it takes for half of a sample of a radioactive substance to change its state

**Ex:** The half-life of radioactive Strontium is 28 days. Write an equation and predict the amount of a 50 gram sample that remains after 100 days.

Use $P(t) = P_0 b^t$: Solve for $b$ when $P_0 = 1$, $P = 0.5$, and $t = 28$.

$$0.5 = 1 b^{28} \Rightarrow b = (0.5)^{\frac{1}{28}} \Rightarrow b \approx 0.9755 \quad \text{Equation:} \quad P(t) = P_0 (0.9755)^t$$

50-gram sample remaining after 100 day: $$P(t) = 50 (0.9755)^{100} \approx 4.185 \text{ grams}$$

Writing a Logistic Function

**Ex:** Find a logistic function that satisfies the given conditions: Initial value = 40; limit to growth = 100; passes through the point $(1,10)$.

$$f(x) = \frac{c}{1 + a \cdot b^x} \quad c = 100, \ a = 40 \quad (x, y) = (1,10) \quad b = ?$$

$$10 = \frac{100}{1 + 40b} \Rightarrow 10 + 400b = 100 \Rightarrow 400b = 90 \Rightarrow b = \frac{9}{40}$$

$$f(x) = \frac{100}{1 + 40 \cdot \left(\frac{9}{40}\right)^x}$$

You Try: Complete the table.

<table>
<thead>
<tr>
<th>Isotope</th>
<th>Half-Life (years)</th>
<th>Initial Quantity</th>
<th>Amount After 1000 Years</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{14}$C</td>
<td>5730</td>
<td>5 grams</td>
<td></td>
</tr>
<tr>
<td>$^{14}$C</td>
<td>5730</td>
<td></td>
<td>0.7 gram</td>
</tr>
<tr>
<td>$^{226}$Ra</td>
<td>1600</td>
<td>100 grams</td>
<td></td>
</tr>
</tbody>
</table>

QOD: Explain how to determine if an exponential function is a growth or decay model.
Syllabus Objective: 9.1 – The student will sketch the graph of an exponential, logistic or logarithmic function. 9.2 – The student will evaluate exponential or logarithmic expressions. 9.5 – The student will graph the inverse of an exponential or logarithmic function.

Review: Solve the following for x.
1. $10^x = 1000$  
   $x = 3$
2. $10^x = 0.01$  
   $x = -2$
3. $10^x = 100\sqrt{10}$  
   $x = 5/2$
4. $10^x = \frac{1}{\sqrt{100}}$  
   $x = -\frac{2}{3}$

Exploration: Use your calculator to write 1997 as a power of 10.

Note: $1000 = 10^3$, $10000 = 10^4$
Try $10^{3.1} \approx 1258.9$, $10^{3.2} \approx 1584.9$, $10^{3.3} \approx 1995.3$ (close!)

Now use your calculator to find $\log(1997)$. $\log(1997) \approx 3.300378$ (Compare to above)

Common Logarithm: Given a positive number $p$, the solution to $10^x = p$ is called the base-10 logarithm of $p$, expressed as $x = \log_{10} p$ , or simply as $x = \log p$ . (When no base is specified, it is understood to be base 10.)

A LOGARITHM IS AN EXPONENT.

Logarithm (base $b$): $y = \log_b x$ (Read as “log base $b$ of $x$.”) for $x > 0$, $b > 0$ and $b \neq 1$ if and only if $x = b^y$.

Note: You cannot take the log of a negative number!

Ex: Rewrite each equation (exponential form) to logarithmic form.
1. $10^3 = 1000$  
   Base ($b$) = 10  
   Exponent = 3  
   log$_{10}$ 1000 = 3 or log$_{10}$ 1000 = 3
2. $2^4 = 16$  
   Base ($b$) = 2  
   Exponent = 4  
   log$_2$ 16 = 4

Ex: Rewrite each equation (logarithmic form) to exponential form.
1. $\log_5 125 = 3$  
   Base ($b$) = 5  
   Exponent = 3  
   $5^3 = 125$
2. $\log_{10} 100,000 = 5$  
   Base ($b$) = 10  
   Exponent = 5  
   $10^5 = 100,000$
Evaluating Logarithms: a logarithm is an exponent

**Ex:** Evaluate the logarithms.

1. \( y = \log_3 81 \) \( 3^y = 81 \) \( \boxed{y = 4} \)
2. \( f\left(\frac{1}{16}\right), \ f(x) = \log_4 x \) \( f\left(\frac{1}{16}\right) = \log_4\left(\frac{1}{16}\right) \) \( 4^y = \frac{1}{16} \) \( \boxed{y = -2} \)
3. \( y = \log_5 1 \) \( 5^y = 1 \) \( \boxed{y = 0} \)
4. \( f(2.2), \ f(x) = \log x \) \( f(x) = \log 2.2 \) Calculator: \( \boxed{f(x) \approx 0.342} \)

Graphing Logarithmic Functions: a logarithmic function is the inverse of an exponential function

**Ex:** Use a table of values to sketch the graph of \( y = \log_3 x \). Discuss the characteristics of the graph and compare the graph to the graph of \( y = 3^x \).

Note: To create the table, it is helpful to rewrite the function as \( 3^y = x \) and choose values for \( y \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
x & 9 & 3 & 1 & 1/3 & 1/9 \\
\hline
y & 2 & 1 & 0 & -1 & -2 \\
\hline
\end{array}
\]

Domain: \((0, \infty)\) Range: \((\infty, \infty)\) Intercepts: \(x - \text{int}: (1,0)\) Asymptote: \(x = 0\)

Increasing: \((0, \infty)\) End Behavior: \(\lim_{x \to 0^-} f(x) = -\infty\) \(\lim_{x \to \infty} f(x) = \infty\)

The function \( y = \log_3 x \) is the inverse of \( y = 3^x \), so its graph is the reflection of \( y = 3^x \) over the line \( y = x \).

Transformations of \( y = \log_b x \):

- Vertical Stretch: \(a, |a| > 1\)
- Horizontal Stretch: \(a, |a| < 1\)
- Reflection over \(x\)-axis: \(-a\)
- Reflection over \(y\)-axis: \(-x\)
- Horizontal Translation: \(h\)
- Vertical Translation: \(k\)
Ex: Describe the transformations used to graph the function \( y = -\log_3(x + 2) - 5 \). Then sketch the graph.
Reflect over \( x \)-axis; shift left 2 units; shift down 5 units

\[
\begin{align*}
\text{Note: Graph in red is the graph of } y &= \log_3 x.
\end{align*}
\]

Properties of Logarithms:
- \( \log_a 1 = 0 \) because \( a^0 = 1 \)
- \( \log_a a = 1 \) because \( a^1 = a \)
- \( \log_a a^x = x \) because \( a^x = a^x \) or \( a^{\log_a x} = x \) (property of inverses)
- If \( \log_a x = \log_a y \), then \( x = y \)

Natural Logarithmic Function: logarithm base \( e \), \( \log_e x \), written \( y = \ln x \)

Notations:
1. \( \log x \) is used to represent \( \log_{10} x \) (common logarithm)
2. \( \ln x \) is used to represent \( \log_e x \) (natural logarithm)

Evaluating Logarithmic Expressions:

Ex: Evaluate \( \log(10^{-2.5}) \) and \( 10^{\log 4.1} \).
\[
\begin{align*}
\log(10^{-2.5}) & \quad \text{Let } y = \log(10^{-2.5}). \text{ Rewrite in exponential form: } 10^y = 10^{-2.5} \quad \boxed{y = -2.5} \\
10^{\log 4.1} & \quad \text{Let } y = 10^{\log 4.1}. \text{ Rewrite in logarithmic form: } \log y = \log 4.1 \quad \boxed{y = 4.1}
\end{align*}
\]

Note: By the properties of inverses, we could have evaluated the above examples without rewriting using the following: \( \log_a a^x = x \) and \( a^{\log_a x} = x \)
Solving Logarithmic Equations:

**Ex:** Solve the equations for \( x \).

1. \( \log_7 x^2 = \log_7 36 \)  
   The bases are equal, so \( x^2 = 36 \) \( \Rightarrow x = \pm 6 \).  
   Note: Both solutions work in the original equation.

2. \( \log_9 3 = x \)  
   Rewrite exponentially: \( 9^x = 3 \) \( \Rightarrow x = \frac{1}{2} \)

3. \( \ln e^4 = x \)  
   Rewrite exponentially: \( e^x = e^4 \) \( \Rightarrow x = 4 \)  
   Note: Remember that the base of \( \ln \) is \( e \).

**You Try:** Sketch the graph of function on the same grid with its inverse. \( f(x) = \ln x \)

**QOD:** Can you evaluate the log of a negative number? Explain.
Syllabus Objective: 9.3 – The student will apply the properties of logarithms to evaluate expressions, change bases, and re-express data.

Exploration: Use your calculator to find \( \log 2 \) and \( \log 3 \). Evaluate the following logarithms on your calculator, then speculate how you could calculate them using only the values of \( \log 2 \) and/or \( \log 3 \).

\[
\begin{align*}
\log 2 & \approx 0.301 \quad \text{and} \quad \log 3 \approx 0.477 \\
1. \quad \log 6 &= \log 6 \approx 0.778 \quad 0.778 = 0.301 + 0.477 = \log 2 + \log 3 \\
2. \quad \log 8 &= \log 8 \approx 0.903 \quad 0.903 = 3 \cdot 0.301 = 3 \log 2 \\
3. \quad \log \frac{2}{3} &= \log \frac{2}{3} = -0.176 \quad -0.176 = 0.301 - 0.477 = \log 2 - \log 3 \\
\end{align*}
\]

Recall: Properties of Exponents

\[
b^x \cdot b^y = b^{x+y} \quad \frac{b^x}{b^y} = b^{x-y} \quad (b^x)^y = b^{xy}
\]

Because a logarithm is an exponent, the rules are the same!

Properties of Logarithms:

\[
\log_b (RS) = \log_b R + \log_b S \\
\log_b \left( \frac{R}{S} \right) = \log_b R - \log_b S \\
\log_b (R^c) = c \cdot \log_b R
\]

Ex: Use the properties of logarithms to expand the following expressions.

\[
\begin{align*}
1. \quad \log (5x) &= \log (5) + \log (x) \\
2. \quad \ln \left( \frac{xy}{z} \right) &= \ln (x) + \ln (y) - \ln (z) \\
3. \quad \log_2 \left( \frac{x}{\sqrt{x^2 + 1}} \right) &= \log_2 x - \log (x^2 + 1)^{\frac{1}{2}} = \log_2 x - \frac{1}{2} \log (x^2 + 1) \\
4. \quad \log (200x^2) &= \log (100 \cdot 2) + 2 \log x = \log 100 + \log 2 + 2 \log x = 2 + \log 2 + 2 \log x
\end{align*}
\]
Ex: Use the properties of logarithms to condense the following expressions.

1. \( \log_3 x + \log_3 4 \)
   \( \log_3 x + \log_3 4 = \log_3 (4x) \)

2. \( \log x - 3 \log (x + 1) \)
   \( \log x - 3 \log (x + 1) = \log x - \log ((x + 1)^3) = \log \frac{x}{(x + 1)^3} \)

3. \( 2 \ln x - \ln (x + 1) - \ln (x - 1) \)
   \[ 2 \ln x - \ln (x + 1) - \ln (x - 1) = \ln x^2 - (\ln (x + 1) + \ln (x - 1)) = \ln \frac{x^2}{(x + 1)(x - 1)} = \ln \frac{x^2}{x^2 - 1} \]

Evaluating Logarithmic Expressions with Base \( b \)

Let \( y = \log_b x \). Rewrite in exponential form: \( b^y = x \).

Take the log of both sides. \( \log b^y = \log x \)

Use the properties of logs to solve for \( y \). \( y \log b = \log x \Rightarrow y = \frac{\log x}{\log b} \)

Note: This will work for a logarithm of any base, including the natural log.

**Change of Base Formula:** \( \log_b a = \frac{\log a}{\log b} = \frac{\ln a}{\ln b} \)

Ex: Evaluate the expression \( \log_3 8 \).

By the change of base formula, \( \log_3 8 = \frac{\log 8}{\log 5} \). Using the calculator, \( \frac{\log 8}{\log 5} \approx 1.292 \)

Ex: Evaluate the expression \( \log_{64} 2 \).

By the change of base formula, \( \log_{64} 2 = \frac{\ln 2}{\ln 64} \). Using the calculator, \( \frac{\ln 2}{\ln 64} \approx 0.167 \)

Note – This did NOT require the use of a calculator! We know that \( 64^{\frac{1}{6}} = 2 \). So \( \log_{64} 2 = \frac{1}{6} \)
Graphing Logarithmic Functions on the Calculator

Ex: Graph \( y = \log_3 x \) and \( y = -\log_3 (x + 2) - 5 \) on the same grid on the graphing calculator.

We cannot type in a log base 3 into the calculator, so we must rewrite the functions using the change of base formula.

\[
y_1 = \log_3 x \implies y_1 = \frac{\log x}{\log 3} \quad y_2 = -\log_3 (x + 2) - 5 \implies y_2 = -\frac{\log (x + 2)}{\log 3} - 5
\]

Caution: The graph created by the calculator is misleading at the asymptote!

You Try: Expand the expression using the properties of logarithms. \( \ln \left( \frac{x^3}{yz} \right) \).

QOD: When is it appropriate to use the change of base formula? Explain how to evaluate a logarithm of base \( b \) without the change of base formula.
Precalculus Notes: Unit 3 – Exponential and Logarithmic Functions

Syllabus Objectives: 9.4 – The student will solve exponential, logarithmic and logistic equations and inequalities. 9.6 – The student will compare equivalent logarithmic and exponential equations.

Strategies for Solving an Exponential Equation:
- Rewrite both sides with the same base
- Take the log of both sides after isolating the exponential

Ex: Solve the following exponential equations.
1. \(4^x = 32\)  Rewrite with base 2: \((2^2)^x = 2^5 \Rightarrow 2^{2x} = 2^5\)  
Both sides have the same base, so the exponents must be equal: \(2x = 5 \Rightarrow x = \frac{5}{2}\)
2. \(8^{3x} = 360\)  We cannot rewrite both sides with the same base, so take the log of both sides.  
\[ \log_8 360 = 3x \cdot \log_8 8 = 3x \]  
Solve for \(x\): \(x = \frac{\log 360}{3 \log 8} \Rightarrow x \approx 0.944\)
3. \(7 - 2e^x = 5\)  Isolate the exponential: \(-2e^x = -2 \Rightarrow e^x = 1\)  
Take the natural log of both sides: \(\ln e^x = \ln 1 \Rightarrow x \ln e = \ln 1 \Rightarrow x = \frac{\ln 1}{\ln e} \Rightarrow x = 0\)

Note: We could have determined that \(x = 0\) immediately using the equation \(e^x = 1\).

Strategies for Solving a Logarithmic Equation:
- Condense any logarithms with the same base using the properties of logs
- Rewrite the equation in exponential form
- Check for extraneous solutions

Ex: Solve the following logarithmic equations.
1. \(\ln x - \ln 5 = 0\)  Condense: \(\ln \left(\frac{x}{5}\right) = 0\)
Rewrite in exponential form: \(e^0 = \frac{x}{5}\)  
Solve for \(x\): \(1 = \frac{x}{5} \Rightarrow x = 5\)  
Check: \(\ln 5 - \ln 5 = 0\) \(\Box\)
2. \(\log_x 625 = 4\)  Rewrite in exp. form: \(x^4 = 625\)
Take the log of both sides: \(\ln x^4 = \ln 625 \Rightarrow 4 \ln x = \ln 625 \Rightarrow \ln x = \frac{\ln 625}{4}\)
Rewrite in exponential form: \(x = e^{\frac{\ln 625}{4}} \Rightarrow x = 5\)  
Note: We could have determined that \(x = 5\) immediately using the equation \(x^4 = 625\), by taking the 4\(^{th}\) root of each side.
3. \( \log_3 (x + \log_3 (x - 8)) = 2 \)  
Condense: \( \log_3 \left( x \left( x - 8 \right) \right) = 2 \)

Rewrite in exponential form:  \( 3^2 = x \left( x - 8 \right) \)

Solve for \( x \):  \( 9 = x^2 - 8x \Rightarrow x^2 - 8x - 9 = 0 \Rightarrow (x - 9)(x + 1) = 0 \Rightarrow x = -1, 9 \)

Check:  \( x = -1 \): \( \log_3 \left( -1 \right) + \log_3 \left( -1 - 8 \right) \neq 2 \) (can’t take the log of a negative!)

\( x = 9 \): \( \log_3, 9 + \log_3, (9 - 8) = 2 + \log_3, 1 = 2 + 0 = 2 \) \( \checkmark \)

\( x = 9 \)

\textbf{Note:} You must check every possible solution for extraneous solutions. All negative answers are not necessarily extraneous!

\textbf{Ex:} Solve the equation \( \log_5 \left( 4 - x \right) = 1 \).

Rewrite in exponential form:  \( 5^1 = 4 - x \)  
Solve for \( x \): \( x = -1 \)

Check:  \( \log_5 \left( 4 - (-1) \right) = \log_5 5 = 1 \) \( \checkmark \)

\( x = -1 \)

\textbf{Ex:} Solve the equation \( \log x^2 = 6 \).

Rewrite using properties of logs:  \( 2 \log x = 6 \Rightarrow \log x = 3 \)

Rewrite in exponential form:  \( 10^3 = x \Rightarrow x = 1000 \)

Check:  \( \log \left( 1000^2 \right) = \log \left( \left( 10^3 \right)^2 \right) = \log (10^6) = 6 \) \( \checkmark \)

\textbf{Challenge Problems:} Use your “arsenal” of exponential and logarithmic properties!

\textbf{Ex:} Solve the equation \( 2^{2x} - 6(2^x) - 7 = 0 \).

Rewrite the first term: \( \left( 2^x \right)^2 - 6 \left( 2^x \right) - 7 = 0 \)  
Let \( 2^x = y \)  
Solve for \( y \): \( (y - 7)(y + 1) = 0 \) \( \Rightarrow y = -1, 7 \)  
Use \( y \) to solve for \( x \):  \( 2^x = y \)

\( y = -1 \): \( 2^x = -1 \) (no solution)

\( y = 7 \): \( 2^x = 7 \Rightarrow \log 2^x = \log 7 \Rightarrow x \log 2 = \log 7 \Rightarrow x = \frac{\log 7}{\log 2} \approx 2.807 \)

\( x = 2.807 \)

\textbf{Ex:} Solve the equation \( 4^{3x} = 7^{x-1} \).

Take the log of both sides: \( \ln 4^{3x} = \ln 7^{x - 1} \)  
Rewrite: \( 3x \cdot \ln 4 = \left( x - 1 \right) \ln 7 \)

Solve for \( x \): \( 3x \cdot \ln 4 = x \ln 7 - \ln 7 \Rightarrow (3 \ln 4)x - (\ln 7)x = -\ln 7 \Rightarrow x (3 \ln 4 - \ln 7) = -\ln 7 \)

\( x = \frac{-\ln 7}{3 \ln 4 - \ln 7} \Rightarrow x \approx -0.879 \)  
Check \( \checkmark \)
You Try: Solve the equation \( \frac{200}{1 + e^{-x}} = 150 \).

QOD: Compare and contrast the methods for solving exponential and logarithmic equations.
Syllabus Objective: 9.7 – The student will solve application problems involving exponential and logarithmic functions.

Newton’s Law of Cooling: The temperature $T$ of an object at time $t$ is $T(t) = T_s + (T_0 - T_s)e^{-kt}$, where $T_s$ is the surrounding temperature and $T_0$ is the initial temperature of the object.

Ex: A 350°F potato is left out in a 70°F room for 12 minutes, and its temperature dropped to 250°F. How many more minutes will it take to reach 120°F?

Solve for $k$ using the given information: $T(t) = T_s + (T_0 - T_s)e^{-kt}$

$250 = 70 + (350 - 70)e^{-12k} \Rightarrow 180 = 280e^{-12k} \Rightarrow \frac{9}{14} = e^{-12k}$

$\ln\left(\frac{9}{14}\right) = \ln(e^{-12k}) \Rightarrow \frac{9}{14} = -12k \Rightarrow k = \frac{-12}{\ln\left(\frac{9}{14}\right)} \approx 0.0368$

Use $k$ to solve for $t$:

$120 = 70 + (350 - 70)e^{-0.0368t} \Rightarrow \frac{50}{28} = e^{-0.0368t} \Rightarrow \ln\left(\frac{5}{28}\right) = -0.0368t \Rightarrow t \approx 46.8$

It takes about 46.8 minutes for the potato to cool to 120°F. This is 46.8 – 12 = 35.8 minutes longer.

Formula for Compound Interest: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$A = \text{balance} \quad r = \text{annual interest rate} \quad P = \text{principal} \quad t = \text{time in years} \quad n = \text{number of times interest is compounded each year}$

Interest Compounded Continuously: $A = Pe^{rt}$

Ex: How long will it take for an investment of $2,000 at 6% compounded semi-annually to reach $5000?

$A = P\left(1 + \frac{r}{n}\right)^{nt}$

$5000 = 2000\left(1 + \frac{0.06}{2}\right)^{2t} \Rightarrow \frac{5}{2} = 1.03^{2t} \Rightarrow \ln\left(\frac{5}{2}\right) = 2t \ln 1.03 \Rightarrow \frac{5}{2} = 2t \cdot \ln 1.03$

$t = \frac{\ln\left(\frac{5}{2}\right)}{2 \ln 1.03} \approx 15.499 \quad \text{It will take about 15.5 years.}$
**Ex:** How long will it take for an investment of $2,000 at 6% compounded continuously to reach $5,000?

\[ A = Pe^{rt} \quad 5000 = 2000e^{0.06t} \Rightarrow \ln \frac{5}{2} = \ln e^{0.06t} \Rightarrow 0.06t = \ln \frac{5}{2} \Rightarrow t = \frac{\ln \frac{5}{2}}{0.06} \approx 15.27 \]

It will take about 15.27 years.

**Annual Percentage Yield (APY):** the rate, compounded annually \((t = 1)\), that would yield the same return

For \( A = P \left( 1 + \frac{r}{n} \right)^n \), \( \text{APY} = \left( 1 + \frac{r}{n} \right)^n - 1 \)

**Ex:** An amount of $2,400 is invested for 8 years at 5% compounded quarterly. What is the equivalent APY?

\[ \text{APY} = \left( 1 + \frac{r}{n} \right)^n - 1 = \left( 1 + \frac{0.05}{4} \right)^4 - 1 \approx 0.0509 \approx 5.09\% \]

**You Try:** Determine the amount of money that should be invested at 9% interest compounded monthly to produce a balance of $30,000 in 15 years.

**QOD:** Why is using the annual percentage yield a more “fair” way to compare investments?