Syllabus Objective: 2.9 – The student will sketch the graph of a polynomial, radical, or rational function.

Polynomial Function: a function that can be written in the form

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$, where n is a nonnegative integer and $a_n \neq 0$

Note: This is called *Standard Form* – when the exponents of x descend

The graph of a polynomial function is a continuous curve with smooth round turns.

Degree of a Polynomial Function: the largest exponent, $\boxed{n}$, of x

Leading Coefficient of a Polynomial Function: the coefficient of the first term when written in standard form, $\boxed{a_n}$

Constant Term: the numerical term, $\boxed{a_0}$

Classifying Polynomial Functions

I. Number of Terms

- 1 term = **monomial**
- 2 terms = **binomial**
- 3 terms = **trinomial**
- 4 or more terms = **polynomial**

II. Degree

Degree	**Name (Degree)**	**Standard Form**	**Example**	**Classification of Example**
0	**Constant**	$f(x) = a$	$y = -3$	Constant Monomial
1	**Linear**	$f(x) = ax + b$	$y = 2x$	Linear Monomial
2	**Quadratic**	$f(x) = ax^2 + bx + c$	$y = -x^2 + 5x + 7$	Quadratic Trinomial
3	**Cubic**	$f(x) = ax^3 + bx^2 + cx + d$	$y = 5x^3 - 2$	Cubic Binomial
4	**Quartic**	$f(x) = ax^4 + bx^3 + cx^2 + dx + e$	$y = x^4 - x^2 + 9x + 1$	Quartic Polynomial
5	**Quintic**	$f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$	$y = x^5$	Quintic Monomial

Ex1: Which of the following are polynomial functions? State the degree and the leading coefficient or explain why it is not a function.

a) $g(x) = 3x^{-8} + 17$ NOT a polynomial functions because $n = -8$.

b) $h(x) = 13x^2 - 7x^5 + 11$ Degree = 5, Leading Coefficient = −7 (quintic trinomial)

Linear Functions

- A line in the Cartesian plane is a **linear function** if and only if it is a *slant* line (not vertical because it fails the vertical line test; not horizontal because it is a constant function).
- The **slope**, or **rate of change** of a linear function is constant.

$$\text{Average Rate of Change} = \frac{f(b)-f(a)}{b-a} = m = \frac{\text{rise}}{\text{run}} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

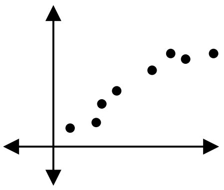
Ex2: Write an equation of the linear function that has $f(-2) = 3$ and $f(3) = -4$.

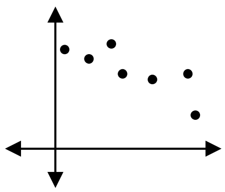
Find the slope of the line. $m = \dfrac{f(3)-f(-2)}{3-(-2)} = \dfrac{-4-3}{3+2} = -\dfrac{7}{5}$

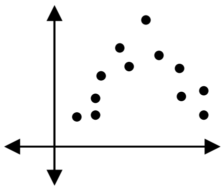
Write equation using point-slope form. $\boxed{y - 3 = -\frac{7}{5}(x+2)}$ or $\boxed{y + 4 = -\frac{7}{5}(x-3)}$

Modeling with Linear Functions

Correlation: Positive Negative Relatively No Linear Correlation



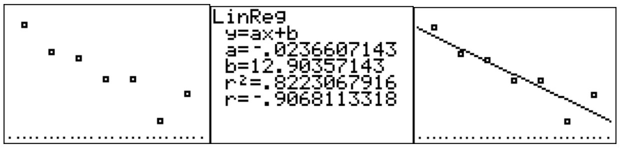




Slope: Positive Negative no line

Ex3: Use the data to write a linear model for the Women's Olympic 100 meter times below. Describe the correlation and predict the time in 2010.

Year (19___)	48	56	64	72	80	88	96
Time (sec)	11.9	11.5	11.4	11.1	11.1	10.5	10.9



```
LinReg
 y=ax+b
 a=-.0236607143
 b=12.90357143
 r²=.8223067916
 r=-.9068113318
```

$\boxed{y = -0.02x + 12.9}$

2010: $y(120) = -0.02(120) + 12.9 = \boxed{10.5 \text{ sec}}$

Quadratic Function: a polynomial function of degree 2

- Graph is u-shaped, called a **parabola**
- Parabolas are symmetric with respect to a line called the **axis of symmetry**
- Parabolas have a **vertex**, which is the point on the parabola where it intersects the axis of symmetry, and is the maximum or minimum of the quadratic function
- General Form: $f(x)=ax^2+bx+c$ Axis of Symmetry: $x=-\frac{b}{2a}$
  - o x-intercepts can be found using the quadratic formula: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
- Vertex Form: $f(x)=a(x-h)^2+k$ Vertex: (h,k) Axis of Symmetry: $x=h$
  - o In both forms, if $a>0$, the parabola opens UP; if $a<0$, the parabola opens DOWN

Graphing a Parabola

Ex4: Write the equation in vertex form. Find the vertex and axis of symmetry. Then graph the function. $f(x)=2x^2+4x-1$

Complete the square to write in vertex form:

$$\frac{y}{2}=\frac{2x^2}{2}+\frac{4x}{2}-\frac{1}{2} \Rightarrow \frac{y}{2}=x^2+2x+\boxed{\left(\frac{2}{2}\right)^2}-\frac{1}{2}-\boxed{\left(\frac{2}{2}\right)^2}$$

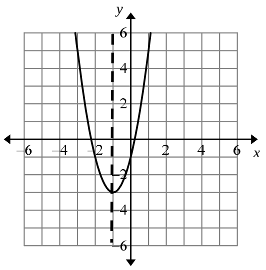
$$\frac{y}{2}=x^2+2x+1-\frac{1}{2}-1 \Rightarrow \frac{y}{2}=(x+1)^2-\frac{3}{2} \Rightarrow \boxed{f(x)=2(x+1)^2-3}$$

Alternate Method:

$$f(x)=2x^2+4x-1 \Rightarrow f(x)=2(x^2+2x)-1$$

$$f(x)=2\left(x^2+2x+\boxed{1}\right)-1-2\left(\boxed{1}\right) \Rightarrow \boxed{f(x)=2(x+1)^2-3}$$

Vertex: $(-1,-3)$ Axis of Symmetry: $x=-1$



Writing the Equation of a Parabola

- Substitute the vertex for (h,k) in vertex form.
- Substitute the given point for (x,y) and solve for a.

Ex5: Write an equation for the parabola with vertex $(-3,-4)$ that passes through the point $(-4,-6)$.

$(h,k)=(-3,-4) \qquad f(x)=a(x+3)^2-4$

$(x,y)=(-4,-6) \qquad -6=a(-4+3)^2-4 \Rightarrow -6=a-4 \Rightarrow a=-2$

$$\boxed{f(x)=-2(x+3)^2-4}$$

Average Rate of Change (recall): $\dfrac{f(b)-f(a)}{b-a}$

Ex6: Find the average rate of change of $f(x)=x^2$ from $x=2$ to $x=4$.

Ave Rate of Change = $\dfrac{f(4)-f(2)}{4-2}=\dfrac{16-4}{2}=\boxed{6}$

Vertical Free-Fall Motion: formula for the position of an object in free fall

$$s(t)=-\frac{1}{2}gt^2+v_0t+s_0 \qquad g\approx 32\text{ ft/sec}^2\approx 9.8\text{ m/sec}^2\text{ (on Earth)}$$

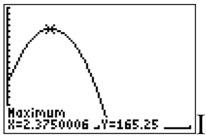
s = position $\quad g$ = acceleration due to gravity $\quad v_0$ = initial velocity $\quad s_0$ = initial position

Ex7: A flare is shot straight up from a ship's bridge 75 feet above the water with an initial velocity of 76 ft/sec. How long does it take to reach its maximum height? How long does it take to reach 163 feet?

$$s_0=75 \qquad v_0=76 \qquad g\approx 32\text{ ft/sec}^2$$

$$s(t)=-\frac{1}{2}(32)t^2+76t+75 \Rightarrow s(t)=-16t^2+76t+75$$

The flare reaches it max height at the vertex of the parabola.



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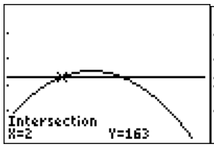
Maximum
X=2.3750006  Y=165.25
```

It takes $\boxed{2.375\text{sec}}$ for the flare to reach its maximum height.

▽ Note: This is NOT the graph of the *path* of the flare. The graph relates the position of the flare to time.

The flare reaches 163 feet when $163=-16t^2+76t+75$.



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Intersection
X=2          Y=163
```
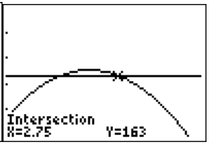



```
Intersection
X=2.75       Y=163
```

It reaches 163 feet at $\boxed{2\text{sec \& }2.75\text{ sec}}$

$$163 = -16t^2 + 76t + 75 \Rightarrow 16t^2 - 76t + 88 = 0 \Rightarrow 4t^2 - 19t + 22 = 0$$

Algebraic Method: $4t^2 - 11t - 8t + 22 = 0 \Rightarrow t(4t - 11) - 2(4t - 11) = 0$

$$(t-2)(4t-11) = 0 \Rightarrow t = 2, \frac{11}{4}$$

You Try: Write the function in vertex form. Find the vertex and axis of symmetry. Then graph.

$f(x) = -2x + x^2 - 3$

QOD: Using a table of values, how can you determine whether they have a linear or quadratic relationship?

Syllabus Objectives: 2.1 – The student will graph relations or functions, including real-world applications. 2.9 – The student will sketch the graph of a polynomial, radical, or rational function.

Power Function: a function that can be written in the form $f(x) = k \cdot x^a$, where *k* and *a* are non-zero constants

- k = constant of variation/proportion a = power, $a \neq 0$
- Direct Variation: $a > 0$ Inverse Variation: $a < 0$

Note: Variation is assumed **direct** unless it specifies "inversely".

Direct & Inverse Variation

Ex1: Write the statement as a power function: The area of a circle varies as the square of the radius.

Implied direct variation. A = Area, r = radius $A = k \cdot r^2$

We know that $k = \pi$, so $\boxed{A = \pi r^2}$

Ex2: *y* varies inversely as *q*. $y = 22$ when $q = 6$. Find *q* when $y = 15$.

Inverse variation: $y = kq^{-1}$ or $y = \frac{k}{q}$

Solve for *k* using the given information: $22 = \frac{k}{6} \Rightarrow k = 132$

Use *k* and $y = 15$ to find *q*. $15 = \frac{132}{q} \Rightarrow 15q = 132 \Rightarrow q = \frac{132}{15} = \boxed{\frac{44}{5}}$

Identifying Power Functions

Ex3: Determine if each function is a power function. If yes, state *k* and *a*. (c is a constant.)

a) $f(x) = -0.5x^2$ Yes, $k = -0.5, a = 2$

b) $g(x) = \pi$ No, because $a = 0$

c) $h(x) = 3^x$ No, because the base is not a variable

d) $f(x) = 2x$ Yes, $k = 2, a = 1$

e) $g(x) = \frac{\pi c}{x^3}$ Yes, $k = \pi c, a = -3$

f) $h(x) = c\sqrt{x}$ Yes, $k = c, a = 1/2$

Monomial Function: $f(x) = k$ or $f(x) = kx^a$, where *k* is a constant and *n* is a positive integer

Graphing Power Functions: Teachers – Allow students time to explore the graphs of various power functions on the graphing calculator, including positive, negative, and rational values of *a*.

Ex4: Graph and analyze the function: $f\left(x\right)=x^{-3}$

Domain: $\left(-\infty,0\right)\cup\left(0,\infty\right)$

Range: $\left(-\infty,0\right)\cup\left(0,\infty\right)$

Continuity: Continuous Function. Infinite discontinuity at $x=0$

Increasing/Decreasing: always decreasing

Symmetry: odd; $f\left(-x\right)=-f\left(x\right)$

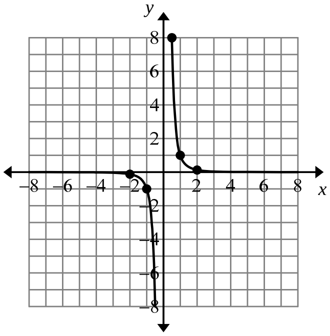
Boundedness: unbounded

Extrema: none

Asymptotes: $y=0,\ x=0$

End Behavior: $\lim\limits_{x\to-\infty} f\left(x\right)=0,\ \lim\limits_{x\to\infty} f\left(x\right)=0$

x	y
-2	$-\frac{1}{8}$
-1	-1
0	Und
1	1
2	$\frac{1}{8}$
$\frac{1}{2}$	8



Modeling with Power Functions

Newton's Law of Cooling

Ex5: The rate at which an object cools varies as the difference between its temperature and the temperature of the surrounding air. When a 270° C steel plate is placed in air that is at 20° C, it is cooling at 50° C per minute. How fast is it cooling when its temperature is 100° C?

R = rate of cooling, T_0 = initial temp, T_S = surrounding temp: $R=k\left(T_0-T_S\right)$

Solve for k: $T_0=270,\ T_S=20,\ R=50$ $50=k\left(270-20\right)\Rightarrow 50=250k\Rightarrow k=\frac{1}{5}$

Answer the question: $R=\frac{1}{5}\left(100-20\right)=\boxed{16^{\circ}\mathrm{C}/\min}$

You Try: The electrical resistance of a wire varies directly as its length and inversely as the square of its diameter. 100 m of a wire with diameter 6 mm has resistance 12 ohms. 80 m of a second wire of the same material has resistance 15 ohms. Find the diameter of the second wire.

QOD: Are all power functions monomial functions? Are all monomial functions power functions? Explain.

Syllabus Objective: 2.9 – The student will sketch the graph of a polynomial, radical, or rational function.

Polynomial Function (recall): $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$, $a_n \neq 0$

Review (if necessary):

Degree	**Name (Degree)**	**Standard Form**	**Example**	**Classification of Example**
0	**Constant**	$f(x) = a$	$y = -3$	Constant Monomial
1	**Linear**	$f(x) = ax + b$	$y = 2x$	Linear Monomial
2	**Quadratic**	$f(x) = ax^2 + bx + c$	$y = -x^2 + 5x + 7$	Quadratic Trinomial
3	**Cubic**	$f(x) = ax^3 + bx^2 + cx + d$	$y = 5x^3 - 2$	Cubic Binomial
4	**Quartic**	$f(x) = ax^4 + bx^3 + cx^2 + dx + e$	$y = x^4 - x^2 + 9x + 1$	Quartic Polynomial
5	**Quintic**	$f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$	$y = x^5$	Quintic Monomial

y-intercept: occurs at $f(0)$. Functions have only 1 *y*-intercept.

Ex1: Describe how to transform the graph. Name the *y*-intercept. $f(x) = -2(x+4)^3 + 7$

"Parent Function" = cubic function $y = x^3$

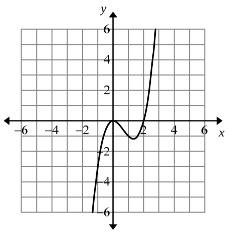
- Vertical stretch by a factor of 2
- Reflect over the *x*-axis
- Shift left 4 units
- Shift up 7 units
- *y*-intercept: $y = -2(0+4)^3 + 7 = -121$

Ex2: Graph the function and list its characteristics. $f(x) = x^3 - 2x^2$

Domain: All Reals Range: All Reals Increasing: $(-\infty, 0) \cup (1, \infty)$ Decreasing: $(0,1)$

Symmetry: None Boundedness: None Extrema: Local Max: $y = 0$ at $x = 0$,

Local Min: $y = -1$ at $x = \frac{4}{3}$ End Behavior: $\lim_{x\to-\infty} = -\infty$; $\lim_{x\to\infty} = \infty$

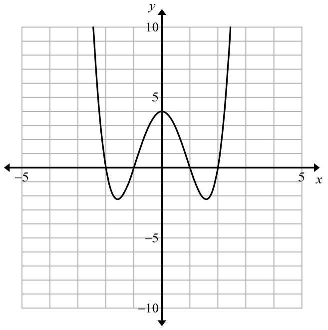


Zeros: $0 = x^2(x-2) \Rightarrow x = 0, 2$

Zeros (*x*-intercepts, Roots) of Polynomial Functions: *n*th-degree polynomials have **at most** $n-1$ extrema and *n* zeros

Ex3: Find the zeros and extrema of the function $f(x)=x^4-5x^2+4$.

Graph:



Zeros: $\{-2,-1,1,2\}$ Extrema: −2.25 (occurs twice), 4

Limit: the value that $f(x)$ **approaches** as *x* **approaches** a given value

Notation: $\lim_{x\to a} f(x)$ "The limit as *x* approaches *a* of $f(x)$." (From both sides.)

$\lim_{x\to a^-} f(x)$ "The limit of $f(x)$ as *x* approaches *a* from the left."

$\lim_{x\to a^+} f(x)$ "The limit of $f(x)$ as *x* approaches *a* from the right."

End Behavior of Polynomial Functions

Odd Degree:

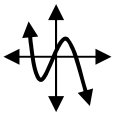
- If the leading coefficient is **positive**, then $\lim_{x\to-\infty} = -\infty$ & $\lim_{x\to\infty} = \infty$.
- If the leading coefficient is **negative**, then $\lim_{x\to-\infty} = \infty$ & $\lim_{x\to\infty} = -\infty$.

Even Degree:

- If the leading coefficient is **positive**, then $\lim_{x\to-\infty} = \infty$ & $\lim_{x\to\infty} = \infty$.
- If the leading coefficient is **negative**, then $\lim_{x\to-\infty} = -\infty$ & $\lim_{x\to\infty} = -\infty$.

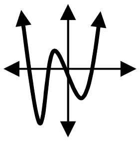
Ex4: Indicate if the degree of the polynomial function shown in the graph is odd or even and indicate the sign of the leading coefficient.

a)



Odd degree; Negative leading coefficient

b)



Even degree; Positive leading coefficient

Zeros of Polynomial Functions

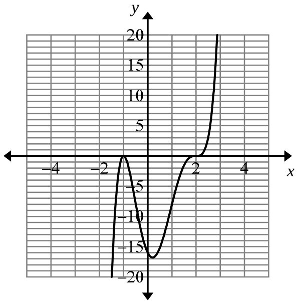
Multiplicity: "repeated" zeros; If a polynomial function *f* has a factor of $(x-c)^m$, and not $(x-c)^{m+1}$, then c is a zero of **multiplicity** *m* of *f*.

- Odd Multiplicity: *f* crosses the *x*-axis at *c*; $f(x)$ changes signs
- Even Multiplicity: f "kisses" or is tangent to the *x*-axis at *c*; $f(x)$ doesn't change signs

Ex5: Sketch the graph of $g(x) = 2(x-2)^3(x+1)^2$. Describe the multiplicity of the zeros.

$x = 2$: multiplicity 3 $\quad$ $x = -1$: multiplicity 2 $\quad$ *y*-intercept: $y = 2(-8)(1) = -16$

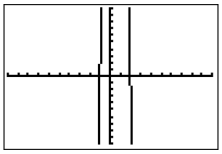
End behavior (odd degree of 5, positive leading coefficient): $\lim\limits_{x\to-\infty} = -\infty$; $\lim\limits_{x\to\infty} = \infty$



Watch for hidden behavior:

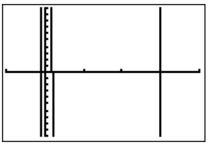
Ex6: Graph the function in a Standard window and find the zeros.

$f(x) = x^3 - 31x^2 + 28x + 60$



$f(x)$ is degree 3, so the end behavior is $\lim\limits_{x\to\infty} = \infty$, so we know it must cross the *x*-axis again to the right.

New window:



Zeros are $x = -1,\ 2,\ 30$

Intermediate Value Theorem (Location Principle): If *a* and *b* are real numbers with $a < b$, and if *f* is continuous on $[a,b]$, then *f* takes on every value between $f(a)$ and $f(b)$.

Therefore, if $f(a)$ and $f(b)$ have opposite signs, then $f(c) = 0$ for some number *c* in $[a,b]$.

Ex7: Use the Intermediate Value Theorem to show $f(x) = 12x^3 - 55x^2 + 18x + 40$ has zeros in these intervals: $[-1, 0]$ and $[1, 2]$.

$[-1, 0]$:
$f(-1) = 12(-1)^3 - 55(-1)^2 + 18(-1) + 40 = -45$ (negative)
$f(0) = 12(0)^3 - 55(0)^2 + 18(0) + 40 = 40$ (positive)

f changes sign (negative to positive) in the interval $[-1, 0]$, so *f* must have at least one zero in $[-1, 0]$.

$[1, 2]$:
$f(1) = 12(1)^3 - 55(1)^2 + 18(1) + 40 = 15$ (positive)
$f(2) = 12(2)^3 - 55(2)^2 + 18(2) + 40 = -48$ (negative)

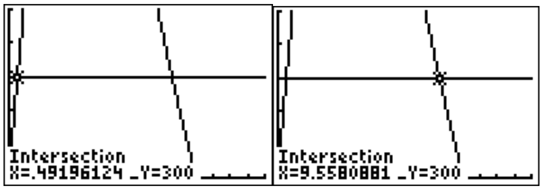
f changes sign (positive to negative) in the interval $[1, 2]$, so *f* must have at least one zero in $[1, 2]$.

Application of Polynomial Functions

Ex8: You cut equal squares from the corners of a 22 by 30 inch sheet of cardboard to make a box with no top. What size squares would need to be cut for the volume to be 300 cubic inches? Define *x* as the length of the sides of the squares. Write a formula for the volume of the box.

$$V(x) = x(22 - 2x)(30 - 2x)$$

Solve the equation $300 = x(22 - 2x)(30 - 2x)$ by graphing.



Intersection
X=.49196124 Y=300

Intersection
X=9.5580881 Y=300

Squares with lengths of approximately 0.492 or 9.558 inches should be cut.

You Try: Sketch the graph of the polynomial function. $f(x) = -\frac{1}{3}(x-4)^2(x+2)(x-5)$

QOD: How many zeros can a function of degree *n* have? Explain your answer.

Syllabus Objectives: 2.2 – The student will calculate the intercepts of the graph of a given relation.

Review: Long Division

$43,581 \div 23$

$$\begin{array}{r}
1894 \\
23\overline{\smash{)}43581} \\
\underline{23} \\
205 \\
\underline{184} \\
218 \\
\underline{207} \\
111 \\
\underline{92} \\
19
\end{array}$$

Remainder is 19, so the quotient is $\boxed{1894\frac{19}{23}}$

Long Division of Polynomials (same process!)

Ex1: Find the quotient. $\left(x^4 - 8x^3 + 11x - 6\right) \div \left(x + 3\right)$

▽ Note: Every term of the polynomial in the dividend must be represented. Since this polynomial is missing an x^2 term, we must include the term $0x^2$.

$$\begin{array}{r}
x^3 - 11x^2 + 33x - 88 \\
x+3\overline{\smash{)}x^4 - 8x^3 + 0x^2 + 11x - 6} \\
\underline{x^4 + 3x^3} \\
-11x^3 + 0x^2 \\
\underline{-11x^3 - 33x^2} \\
33x^2 + 11x \\
\underline{33x^2 + 99x} \\
-88x - 6 \\
\underline{-88x - 264} \\
258
\end{array}$$

Solution: $\boxed{x^3 - 11x^2 + 33x - 88 + \frac{258}{x+3}}$

Remainder Theorem: If $f\left(x\right)$ is divided by $x - k$, then the remainder, $r = f\left(k\right)$.

Ex2: Find the remainder without using division for $x^4 - 8x^3 + 11x - 6$ divided by $x + 3$.

$f\left(x\right) = x^4 - 8x^3 + 11x - 6 \qquad r = f\left(-3\right) = \left(-3\right)^4 - 8\left(-3\right)^3 + 11\left(-3\right) - 6 = \boxed{258}$

Note: Compare your answer to the long division problem above.

Factor Theorem: $(x-k)$ is a factor of $f(x)$ if and only if $f(k)=0$.

Ex3: Determine if $x-3$ is a factor of $2x^3-3x^2-5x-12$ without dividing.

Show that $r=f(3)=0$: $2(3)^3-3(3)^2-5(3)-12=0$, so by the Factor Theorem, $x-3$ is a factor of $2x^3-3x^2-5x-12$.

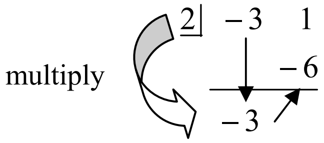
Synthetic Substitution:

Ex4: Find $f(2)$ if $f(x)=-3x^4+x^3-5x^2+6x+1$ using synthetic substitution.

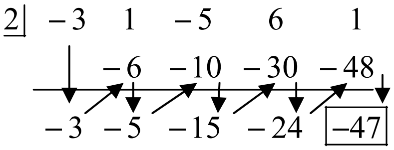
Using the polynomial in standard form, write the coefficients in a row. Put the *x*-value to the upper left.

$$2\,\big|\; -3 \quad 1 \quad -5 \quad 6 \quad 1$$

Bring down the first coefficient, then multiply by the *x*-value.



Add straight down the columns, and repeat.



The number in the bottom right is the value of $f(2)$. *So*: $\boxed{f(2)=-47}$

Ex 5: Find $f(-3)$ if $f(x)=x^5-2x^3+7x^2-11$ using synthetic substitution.

This polynomial function is in standard form, however it is missing two terms. We can rewrite the function as $f(x)=x^5+0x^4-2x^3+7x^2+0x-11$ to fill in the missing terms.

$$\begin{array}{r|rrrrrr} -3 & 1 & 0 & -2 & 7 & 0 & -11 \\ & & -3 & 9 & -21 & 42 & -126 \\ \hline & 1 & -3 & 7 & -14 & 42 & \boxed{-137} \end{array}$$

$$\boxed{f(-3)=-137}$$

This also means that $(-3,-137)$ is an ordered pair that would be a point on the graph. And the remainder when dividing $f(x)$ by $x+3$ is -137.

Recall: $f(x)=a_nx^n+a_{n-1}x^{x-1}+\ldots+a_2x^2+a_1x+a_0$

Possible **Rational** Zeros of a Polynomial Function = $\dfrac{\text{factors of constant }(a_0)}{\text{factors of leading coefficient }(a_n)}$

Ex6: Find the possible rational zeros of $f(x) = x^2 + 7x + 12$

Step 1: The leading coefficient is 1. 1 is the only factor of 1.

Step 2: The constant is 12. All of the factors of 12 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$.

Step 3: List the possible factors - $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1}$, and $\pm\frac{12}{1}$

*If we tested for actual zeros using synthetic substitution we would find that -3 and -4 are zeros.

This also means that 5 could not be a zero, 7 could not be a zero, $\frac{1}{2}$ could not be a zero, ...

<u>Descartes' Rule of Signs</u>: The number of real zeros equals the number of sign changes in $f(x)$ or that number less a multiple of 2. The number of negative real zeros equals the number of sign changes in $f(-x)$ or that number less a multiple of 2.

Ex7: Use Descartes' Rule of Signs to determine the possible number of positive and negative real zeros of the function $f(x) = 3x^3 + 4x^2 - 5x - 2$.

Number of sign changes of $f(x)$: $f(x) = 3x^3 + 4x^2 - 5x - 2$ 1 sign change

Number of sign changes of $f(-x)$: $f(-x) = -3x^3 + 4x^2 + 5x - 2$ 2 sign changes

There is 1 possible positive real zero and 2 or 0 possible negative real zeros.

<u>Upper and Lower Bound Rules</u>: used to determine if there is no zero larger or smaller than a number *c*, if the leading coefficient is POSITIVE $(a_n > 0)$, using synthetic division for $f(x) \div (x - c)$

- Upper Bound: *c* is an **upper bound** for the real zeros of *f* if $c \geq 0$ and every number in the last line of $f(x) \div (x - c)$ using synthetic division has signs that are all nonnegative.
- Lower Bound: *c* is a **lower bound** for the real zeros of *f* if $c \leq 0$ and every number in the last line of $f(x) \div (x - c)$ using synthetic division has signs that are alternately nonnegative and nonpositive.

Ex8: Verify that all of the real zeros of the function $f(x) = 3x^3 + 4x^2 - 5x - 2$ lie on the interval $[-3, 2]$.

$$f(2): \qquad \begin{array}{r|rrrr} 2 & 3 & 4 & -5 & -2 \\ & & 6 & 20 & 30 \\ \hline & 3 & 10 & 15 & 28 \end{array}$$

$c = 2 \geq 0$ and signs are all nonnegative, so $c = 2$ is an **upper bound** of the zeros

$$f(-3): \qquad \begin{array}{r|rrrr} -3 & 3 & 4 & -5 & -2 \\ & & -9 & 15 & -30 \\ \hline & 3 & -5 & 10 & -32 \end{array}$$

$c = -3 \leq 0$ and signs alternate, so $c = -3$ is an **lower bound** of the zeros

Conclusion: All real zeros lie on the interval $[-3, 2]$.

Finding the Real Zeros of a Polynomial:

Ex9: Find all real solutions of the polynomial equation. $f(x) = 3x^3 + 4x^2 - 5x - 2$

Note: There is one sign change, so there is one positive real zero.

Possible Rational Zeros: $\pm\frac{1}{3}, \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{2}{3}$

Note: Remember, if 1 is a zero, $x - 1$ is a factor

Synthetic Substitution (try 1):

$$f(1) = \begin{array}{r|rrrr} 1 & 3 & 4 & -5 & -2 \\ & & 3 & 7 & 2 \\ \hline & 3 & 7 & 2 & \boxed{0} \end{array}$$

1 is a zero, so we can write $f(x) = (x-1)(3x^2 + 7x + 2) = 0$.

Factor the quadratic: $(x-1)(2x+1)(x+2) = 0$ $\quad 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$; $x + 2 = 0 \Rightarrow x = -2$

Solutions: $\boxed{\left\{-2, -\frac{1}{2}, 1\right\}}$

You Try: Find the real zeros of $f(x) = 2x^4 - 7x^3 - 8x^2 + 14x + 8$.

QOD: How is the Factor Theorem connected to the Remainder Theorem?

Syllabus Objectives: 1.7 – The student will solve problems using the Fundamental Theorem of Algebra. 1.8 – The student will calculate the complex roots of a given function.

Fundamental Theorem of Algebra

- A polynomial function of degree *n* has *n* complex zeros (some zeros may repeat)
- Complex zeros come in conjugate pairs
  Review: $a+bi \Rightarrow \text{conjugate} \Rightarrow a-bi$
- A polynomial function with *odd* degree and real coefficients has at least one real zero

Teacher Note: Have students discuss "why".

Ex1: Find all the zeros of the function and write the polynomial as a product of linear factors.
$f(x)=x^4-6x^3+10x^2-6x+9$

Step One: Check for any rational zeros.

Possible rational zeros: $\dfrac{\text{factors of 9}}{\text{factors of 1}}=\pm\dfrac{1}{1},\pm\dfrac{3}{1},\pm\dfrac{9}{1}$

Use synthetic substitution:

$$\begin{array}{r|rrrrr} 3 & 1 & -6 & 10 & -6 & 9 \\ & & 3 & -9 & 3 & -9 \\ \hline & 1 & -3 & 1 & -3 & 0 \end{array}$$

So 3 is a zero, and $x-3$ is a factor.

Step Two: Rewrite the last line as a polynomial and find the zeros of the polynomial.

$f(x)=(x-3)(x^3-3x^2+x-3)$

$x^3-3x^2+x-3=0$ Factor by grouping: $x^2(x-3)+1(x-3)=0 \Rightarrow (x^2+1)(x-3)=0$

$x^2+1=0$
$\sqrt{x^2}=\sqrt{-1}$
$x=\pm i$

$x-3=0$
$x=3$

Step Three: List the zeros. Check that you have the correct number according to the FTA.

$\boxed{x=3,-i,i}$ The polynomial was degree 4, and there are 4 zeros (3 is repeated).

Note that *i* and –*i* are complex conjugates.

Step Four: Write the polynomial as a product of linear factors.

$\boxed{f(x)=(x-3)(x-3)(x+i)(x-i)}$

Ex2: Find a polynomial function with real coefficients in standard form that has the zeros −2 and $1+2i$.

Write the polynomial as a product of linear factors using the zeros.

$f(x)=(x+2)(x-(1+2i))(x-(1-2i))$

Note: If $1+2i$ is a zero, then its complex conjugate, $1-2i$, must also be a zero.

$$f(x)=(x+2)(x-1-2i)(x-1+2i)$$

Write the polynomial in standard form. $=(x+2)\left(x^2-2x+5\right)$

$$\boxed{f(x)=x^3+x+10}$$

<u>Irreducible Quadratic Factor</u>: A quadratic factor of a polynomial function is *irreducible over the reals* if it has real coefficients but no real zeros.

Ex3: Write the polynomial as the product of factors that are irreducible over the reals. Then write the polynomial in completely factored form. $f(x)=3x^5+x^4-3x^3-x^2-60x-20$

Step One: Check for any rational zeros.

Possible rational zeros: $\pm\frac{1}{1},\pm\frac{2}{1},\pm\frac{4}{1},\pm\frac{5}{1},\pm\frac{10}{1},\pm\frac{20}{1},\pm\frac{1}{3},\pm\frac{2}{3},\pm\frac{4}{3},\pm\frac{5}{3},\pm\frac{10}{3},\pm\frac{20}{3}$

Use synthetic substitution:

$$\begin{array}{r|rrrrrr} -\frac{1}{3} & 3 & 1 & -3 & -1 & -60 & -20 \\ & & -1 & 0 & 1 & 0 & 20 \\ \hline & 3 & 0 & -3 & 0 & -60 & 0 \end{array}$$

Step Two: Rewrite the last line as a polynomial and find the zeros of the polynomial.

$$f(x)=\left(x+\frac{1}{3}\right)\left(3x^4-3x^2-60\right)=3\left(x+\frac{1}{3}\right)\left(x^4-x^2-20\right)=(3x+1)\left(x^4-x^2-20\right)$$

$x^4-x^2-20=0 \Rightarrow \left(x^2-5\right)\left(x^2+4\right)=0$ $\quad x^2-5=0 \quad x^2+4=0$

$x=\pm\sqrt{5} \quad x=\pm 2i$

Product of factors irreducible over the reals: $\boxed{f(x)=(3x+1)\left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)\left(x^2+4\right)}$

Completely factored form: $\boxed{f(x)=(3x+1)\left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)(x-2i)(x+2i)}$

Ex4: Use the given zero to find ALL the zeros of the function. Zero = $1+i$;
$f(x)=x^4-2x^3-x^2+6x-6$

Note: By the FTA, we know that $f(x)$ must have 4 zeros.

Since $1+i$ is a zero, we know that $1-i$ (its conjugate) must also be a zero.

Two factors of $f(x)$ are: $(x-(1+i))$ and $(x-(1-i))$

Multiply: $(x-1-i)(x-1+i)=x^2-2x+2$

Use long division:

$$\begin{array}{rl} & x^2-3 \\ x^2-2x+2 & \enclose{longdiv}{x^4-2x^3-x^2+6x-6} \\ & \underline{x^4-2x^3+2x^2} \\ & \quad\quad -3x^2+6x-6 \\ & \quad\quad \underline{-3x^2+6x-6} \end{array}$$

Zeros of $\begin{array}{l} x^2-3=0 \\ x=\pm\sqrt{3} \end{array}$

Zeros of $f\left(x\right)$: $\boxed{\left\{-\sqrt{3},\sqrt{3},1-i,1+i\right\}}$

<u>You Try</u>: Find a polynomial function with real coefficients in standard form that has the zeros 3, and -1 with multiplicity 3.

<u>QOD</u>: Is the polynomial function found in the "You Try" unique? Explain your answer.

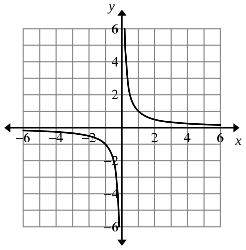
Syllabus Objectives: 1.9 – The student will solve rational equations in one variable. 2.9 – The student will sketch the graph of a polynomial, radical, or rational function.

Rational Function: the ratio of two polynomial functions

$$r(x)=\frac{f(x)}{g(x)},\ g(x)\neq 0$$

Domain of a Rational Function: the set of all real numbers except the zeros if its denominator

Recall: Reciprocal Function $f(x)=\frac{1}{x}$



Rational Functions that are Transformations of the Reciprocal Function

Ex1: Describe the transformations on $f(x)=\frac{1}{x}$. State the domain. $g(x)=\frac{-3}{x+1}-2$

1. $g(x)=\frac{-3}{x+1}-2$ Reflect over x-axis
2. $g(x)=\frac{-3}{x+1}-2$ Vertical stretch by 3
3. $g(x)=\frac{-3}{x+1}-2$ Translate left by 1
4. $g(x)=\frac{-3}{x+1}-2$ Translate down 2

Domain: All reals, $x\neq -1$

Using Long Division (used when variable is in the numerator and denominator)

Ex2: Describe the transformations on $f(x)=\frac{1}{x}$. State the domain. $g(x)=\frac{4-3x}{x-5}$

Long division:
$$\begin{array}{r} -3 \\ x-5\overline{)\,-3x+4} \\ \underline{-3x+15} \\ -11 \end{array}$$

$$g(x)=-3-\frac{11}{x-5}$$

1. $g(x)=-3-\frac{11}{x-5}$ Translate down 3
2. $g(x)=-3-\frac{11}{x-5}$ Reflect over x-axis
3. $g(x)=-3-\frac{11}{x-5}$ Vertical stretch by 11
4. $g(x)=-3-\frac{11}{x-5}$ Translate right 5

Recall:

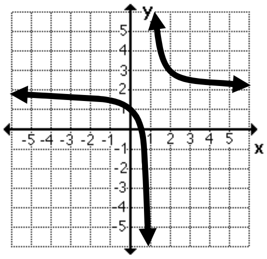
Limit: the value that $f(x)$ **approaches** as *x* **approaches** a given value

Notation: $\lim\limits_{x\to a} f(x)$ "The limit as *x* approaches a of $f(x)$." (From both sides.)

$\lim\limits_{x\to a^-} f(x)$ "The limit of $f(x)$ as *x* approaches a from the left."

$\lim\limits_{x\to a^+} f(x)$ "The limit of $f(x)$ as *x* approaches a from the right."

Ex3: Evaluate the limits based on the graph.



1) $\lim\limits_{x\to 1^-} f(x)$	$\lim\limits_{x\to 1^-} f(x) = -\infty$
2) $\lim\limits_{x\to 1^+} f(x)$	$\lim\limits_{x\to 1^+} f(x) = \infty$
3) $\lim\limits_{x\to -\infty} f(x)$	$\lim\limits_{x\to -\infty} f(x) = 2$
4) $\lim\limits_{x\to \infty} f(x)$	$\lim\limits_{x\to \infty} f(x) = 2$

End Behavior Asymptotes of $f(x) = \dfrac{a_n x^n + \ldots}{b_m x^m + \ldots}$

- If $n < m$: horizontal asymptote $y = 0$
- If $n = m$: horizontal asymptote $y = \dfrac{a_n}{b_n}$ (ratio of the leading coefficients)
- If $n > m$: No horizontal asymptote. End behavior is the function that is the result after performing the division on $f(x)$.

Note: If the *n* is one more than *m*, then the function describing end behavior is **linear**. This is called a **slant asymptote**.

Ex4: Describe the end behavior of the following functions.

1) $y=\dfrac{x^3+3x^2+x+1}{x-1}$ degree of the numerator > degree of the denominator

No horizontal asymptote, no slant asymptote

Long Division: $y=x^2+4x+5+\dfrac{6}{x-1}$

End Behavior: Approaches the graph of the function $\boxed{y=x^2+4x+5}$

2) $y=\dfrac{6x^2}{3(2x+1)^2}$ degree of the numerator = degree of the denominator

Lead coefficient of numerator = 6

Lead coefficient of denominator = 12: $3(2x+1)^2=3(4x^2+4x+1)=12x^2+12x+3$

End Behavior: horizontal asymptote of $y=\dfrac{6}{12}\Rightarrow\boxed{y=\dfrac{1}{2}}$

<u>Graphing Rational Functions</u>: To sketch the graph of a rational function, find the domain, asymptotes, and intercepts. The plot points in each region created by the asymptotes to locate the curve and use the asymptotes to guide the sketching of the curve.

Ex5: Graph $g(x)=\dfrac{x^2-x}{x+1}$.

Domain: All reals, $x\neq-1$ Vertical Asymptote: $x=-1$

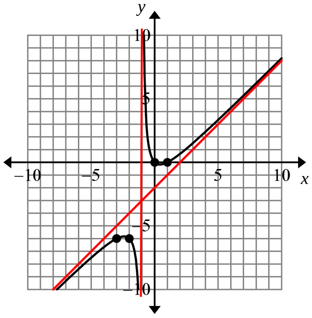
x-intercepts (zeros of numerator): $x^2-x=0\Rightarrow x(x-1)=0\Rightarrow x=0,1$

y-intercepts ($g(0)$): $g(0)=0$

End Behavior: No horizontal asymptote; long division results in $g(x)=x-2+\dfrac{2}{x+1}$

Slant Asymptote: $y=x-2$ Note: $\lim\limits_{x\to\infty}\dfrac{2}{x+1}=0$

Plot a few points to graph the lower part of the curve.

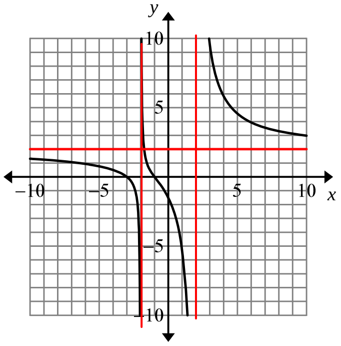


Ex6: Graph $g(x)=\dfrac{2x^2+8x+6}{x^2-4}$.

Domain: All reals, $x \neq -2, 2$ Vertical Asymptotes: $x=-2,\ x=2$

x-intercepts (zeros of numerator): $2\left(x^2+4x+3\right)=0 \Rightarrow 2(x+3)(x+1)=0 \Rightarrow x=-3,-1$

y-intercepts ($g(0)$): $g(0)=-\dfrac{3}{2}$ End Behavior: horizontal asymptote $y=2$



Removable Discontinuity: a rational function has a "hole" in the graph if one or more of the linear factors in the denominator can be cancelled with one of the factors in the numerator.

Ex7: Sketch the graph of $g(x)=\dfrac{2x^2-18}{x^2+3x}$

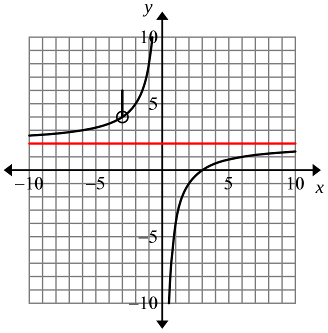
Domain: All reals, $x \neq -3, 0$

Vertical Asymptotes: $g(x)=\dfrac{2\left(x^2-9\right)}{x(x+3)}=\dfrac{2\boxed{(x+3)}(x-3)}{x\boxed{(x+3)}}$ $x=0$

Note: $x=-3$ is NOT a vertical asymptote. It is a hole.

x-intercepts (zeros of numerator): $2(x+3)(x-3)=0 \Rightarrow x=-3,3$

y-intercepts ($g(0)$): None End Behavior: horizontal asymptote $y=2$



You Try: Sketch the graph of $y=\dfrac{x^2-x-2}{x-5}$. Describe all relevant features.

QOD: Can a function cross a vertical asymptote? Explain. Can a function cross a horizontal asymptote? Explain.

Syllabus Objective: 1.9 – The student will solve rational equations in one variable.

Solving a Rational Equation:

1. Multiply every term by the LCD to wipe out the fractions.
2. Solve the resulting equation.
3. Check for **extraneous solutions** in the *original* equation, which can result when multiplying by a variable expression.

Ex1: Solve the equation. $\frac{1}{y-2}+\frac{1}{y+2}=\frac{4}{y^2-4}$

Factor to find LCD: $\frac{1}{y-2}+\frac{1}{y+2}=\frac{4}{(y-2)(y+2)}$ LCD = $(y-2)(y+2)$

Multiply by LCD:

$$\cancel{(y-2)}(y+2)\frac{1}{\cancel{(y-2)}}+(y-2)\cancel{(y+2)}\frac{1}{\cancel{(y+2)}}=\cancel{(y-2)(y+2)}\frac{4}{\cancel{(y-2)(y+2)}}$$

Solve the resulting equation: $y+2+y-2=4 \Rightarrow 2y=4 \Rightarrow y=2$

Check for extraneous solutions: $\frac{1}{2-2}+\frac{1}{2+2}=\frac{4}{2^2-4}$

$y=2$ makes one or more of the denominators in the original equation equal zero, so it is extraneous. Therefore, there is NO SOLUTION.

Ex2: Solve the equation: $\frac{2x}{5x+4}=\frac{3}{3x+8}$

This equation is a proportion, so we will cross multiply.

$$2x(3x+8)=3(5x+4)$$
$$6x^2+16x=15x+12$$

Solve the equation by factoring: $6x^2+x-12=0 \Rightarrow (2x+3)(3x-4)=0$ $\boxed{x=-\frac{3}{2},\frac{4}{3}}$

Check for extraneous solutions: neither solution results in a denominator of zero in the original equation

Ex3: Find the solution set of the equation $\frac{3}{x^2-7x+10}+2=\frac{x-4}{x-5}$.

Factor to find LCD: $\frac{3}{(x-5)(x-2)}+2=\frac{x-4}{(x-5)}$ LCD = $(x-5)(x-2)$

Multiply by LCD: $\cancel{(x-5)(x-2)}\frac{3}{\cancel{(x-5)(x-2)}}+2(x-5)(x-2)=\frac{x-4}{\cancel{(x-5)}}\cancel{(x-5)}(x-2)$

Solve the resulting equation:

$$3+2x^2-14x+20=x^2-6x+8 \Rightarrow x^2-8x+15=0$$

$$(x-5)(x-3)=0 \Rightarrow x=3,5$$

Check for extraneous solutions:
$x=3$ does not make one or more of the denominators in the original equation equal zero, so it is a solution. $x=5$ does make one or more of the denominators in the original equation equal zero, so it is an extraneous solution. Solution: $\boxed{x=3}$

You Try: Solve the equation $\frac{x}{x-2}+\frac{3}{x+3}=\frac{4}{x^2+x-6}$.

QOD: Explain two ways you can check your solution graphically when solving a rational equation.

Sample SAT Question(s): Taken from College Board online practice problems.

1. The projected sales volume of a video game cartridge is given by the function $s(p)=\frac{3000}{2p+a}$, where *s* is the number of cartridges sold, in thousands; *p* is the price per cartridge, in dollars; and *a* is a constant. If according to the projections, 100,000 cartridges are sold at \$10 per cartridge, how many cartridges will be sold at \$20 per cartridge?
   (A) 20,000
   (B) 50,000
   (C) 60,000
   (D) 150,000
   (E) 200,000

2. $S=\frac{a}{b}+\frac{c}{d}+\frac{1}{e}$. If $0<a<b<c<d<e$ in the equation given, then the greatest increase in *S* would result from adding 1 to the value of which variable?
   (A) *a*
   (B) *b*
   (C) *c*
   (D) *d*
   (E) *e*

3. If ☼ is defined for all positive numbers *a* and *b* by *a*☼*b* = $\frac{ab}{a+b}$, then 10☼2 =

   (A) $\frac{5}{3}$

   (B) $\frac{5}{2}$

   (C) 5

   (D) $\frac{20}{3}$

   (E) 20

Syllabus Objective: 2.10 – The student will solve and graph inequalities in one variable.

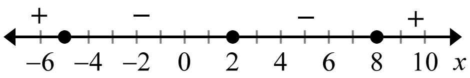
Solving an Inequality:

1. Find the zeros of the corresponding equation and the zeros of the denominators of any rational expressions (called **critical values**).
2. Make a sign chart (choose a test value) to find the correct interval solutions.

Ex1: Solve the inequality $(x+5)(x-2)^4(x-8)<0$

Critical Values: $(x+5)(x-2)^4(x-8)=0 \Rightarrow x=-5,2,8$

Sign Chart:

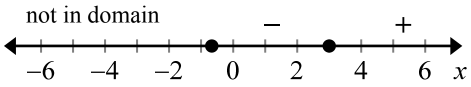


Solution: $\boxed{(-5,2)\cup(2,8)}$

Note: If the inequality was $(x+5)(x-2)^4(x-8)\le 0$, the solution would be $[-5,8]$.

Ex2: Solve the inequality: $\dfrac{\sqrt{3x+2}}{x-3}\ge 0$

Critical Values: $\sqrt{3x+2}=0 \Rightarrow x=-\dfrac{2}{3}$ $\qquad x-3=0 \Rightarrow x=3$



Sign Chart:

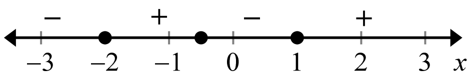
Solution: $\boxed{x=\left\{-\frac{2}{3}\right\}\cup(3,\infty)}$ Note: $x=3$ is not in the domain of the original function.

Ex3: Solve the inequality. $\dfrac{1}{x-1}<-\dfrac{1}{x+2}$

Rewrite the inequality: $\dfrac{1}{x-1}+\dfrac{1}{x+2}<0 \Rightarrow \dfrac{x+2+x-1}{(x-1)(x+2)}<0 \Rightarrow \dfrac{2x+1}{(x-1)(x+2)}<0$

Critical Values:

$$2x+1=0 \qquad x-1=0 \qquad x+2=0$$
$$x=-\frac{1}{2} \qquad x=1 \qquad x=-2$$



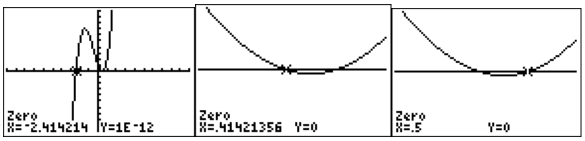
Sign Chart:

Solution: $\boxed{(-\infty,-2)\cup\left(-\frac{1}{2},-1\right)}$

Solving an Inequality Graphically

Ex4: Solve graphically: $2x^3 + 3x^2 - 4x + 1 \le 0$

Find the zeros (watch for hidden zeros!):



```
Zero
X=-2.414214  Y=1E-12
```

```
Zero
X=.41421356  Y=0
```

```
Zero
X=.5         Y=0
```

Note: The 2nd and 3rd graphs were created by zooming in to the local minimum 3 times!

Solution: $\boxed{(-\infty, -2.414] \cup [0.414, 0.5]}$

Application Problem

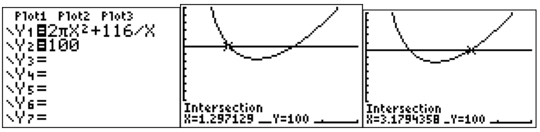
Ex5: A cylindrical can must have a volume of 58 cubic inches. What dimensions can you use if you must keep the surface area below 100 square inches?

Volume: $V = \pi r^2 h = 58$ Surface Area: $SA = 2\pi r^2 + 2\pi rh < 100$

Solve for *h* in the volume equation and substitute into the surface area inequality.

$$h = \frac{58}{\pi r^2} \qquad 2\pi r^2 + 2\pi r \frac{58}{\pi r^2} < 100 \Rightarrow 2\pi r^2 + \frac{116}{r} < 100$$

Solve by graphing:



```
Plot1 Plot2 Plot3
\Y1=2πX²+116/X
\Y2=100
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
Intersection
X=1.297129  Y=100
```

```
Intersection
X=3.1794358 Y=100
```

Solution: Radius must be between 1.297 and 3.179 inches. Height must be between 1.827 and 10.975 inches.

You Try: Solve the inequality $\frac{2x^2 + x - 1}{x^2 - 4x + 4} \le 0$.

QOD: Explain why you need only choose one test point within the intervals created by the critical value.