TAKE IT TO THE MAT

A NEWSLETTER ADDRESSING THE FINER POINTS OF MATHEMATICS INSTRUCTION

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This issue of *Take It to the MAT* focuses on an important, but often difficult, algebraic technique: *completing the square*. Completing the square is usually taught as a method of equation solving, but its applications are more useful. Either way, it is a skill that algebra students must learn.

As an equation solving technique, completing the square is not usually the most efficient method. For instance, completing the square to solve $2x^2+3x-1=0$ is cumbersome, but does get the job done. (See the solution near right.)

$$2x^2+3x-1=0$$
$$2\left(x^2+\tfrac{3}{2}x\right)=1$$
$$2\left(x^2+\tfrac{3}{2}x+\left(\tfrac{3}{4}\right)^2\right)=1+2\left(\tfrac{3}{4}\right)^2$$
$$2\left(x+\tfrac{3}{4}\right)^2=\tfrac{34}{16}$$
$$\left(x+\tfrac{3}{4}\right)^2=\tfrac{17}{16}$$
$$x+\tfrac{3}{4}=\pm\tfrac{\sqrt{17}}{4}$$
$$x=-\tfrac{3}{4}\pm\tfrac{\sqrt{17}}{4}$$

$$ax^2+bx+c=0$$
$$a\left(x^2+\tfrac{b}{a}x\right)=-c$$
$$a\left(x^2+\tfrac{b}{a}x+\left(\tfrac{b}{2a}\right)^2\right)=-c+a\left(\tfrac{b}{2a}\right)^2$$
$$a\left(x+\tfrac{b}{2a}\right)^2=\tfrac{-4a^2c+ab^2}{(2a)^2}=\tfrac{a\left(b^2-4ac\right)}{(2a)^2}$$
$$\left(x+\tfrac{b}{2a}\right)^2=\tfrac{\left(b^2-4ac\right)}{(2a)^2}$$
$$x+\tfrac{b}{2a}=\pm\tfrac{\sqrt{b^2-4ac}}{2a}$$
$$x=-\tfrac{b}{2a}\pm\tfrac{\sqrt{b^2-4ac}}{2a}$$

Since the equation is not factorable, the quadratic formula might be a better approach. However, the reason we have a quadratic formula to use is because of completing the square. If one completed the square on the general quadratic, $ax^2+bx+c=0$, then solved for *x*, the quadratic formula is the result. (See the derivation far right.) Students need to see this derivation, and do it themselves. The quadratic formula is "magic" to some students. It is important that they know the origin of the formula.

When students take second-year algebra and precalculus, they will be expected to know how to complete the square. One of the reasons is that they will study conic sections. For instance, is the equation $9x^2-18x+4y^2+16y-11=0$ that of a circle, an ellipse, a hyperbola, or a parabola? And what are its characteristics?

Computing the discriminant is a quick way to determine *what* conic it is, but that method will not be presented here, as it doesn't tell much else about the figure. The traditional, and probably the best, method is to complete the square on *x* and *y*. The process is shown at right. The end result is that the figure is an ellipse centered at (1, –2). More characteristics of the ellipse can be determined using this form of the equation, if desired.

$$9x^2-18x+4y^2+16y-11=0$$
$$9\left(x^2-2x\right)+4\left(y^2+4y\right)=11$$
$$9\left(x^2-2x+\underline{\underline{\mathbf{1}}}\right)+4\left(y^2+4y+\underline{\mathbf{4}}\right)=11+\underline{\underline{\mathbf{9}}}+\underline{\mathbf{16}}$$
$$9(x-1)^2+4(y+2)^2=36$$

While conic sections are not part of the first-year algebra curriculum, the technique used to change their equations from one form to another—completing the square—is. Additionally, completing the square on a general quadratic derives the quadratic formula. It is very important that students are fluent in this process, for success in first-year algebra and beyond.