AP Calculus Notes: Unit 3 – Derivatives (Part Two)

Syllabus Objective: 2.3 – The student will differentiate the sum, product, and quotient of elementary functions.

Derivatives of Trigonometric Functions:

Teacher Note: Have students graph $y = \sin x$ and $y = \cos x$ in the same viewing window as their derivatives. From the graphs, they should be able to come up with the following.

$$\boxed{\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x}$$

Ex1: Find $f'(x)$ for $f(x) = \dfrac{\cos x}{x^2}$.

Method 1: Use the quotient rule.

$$f'(x) = \frac{x^2(-\sin x) - \cos x(2x)}{x^4} = \frac{x(-x\sin x - 2\cos x)}{x^4} = \boxed{\frac{-x\sin x - 2\cos x}{x^3}}$$

Method 2: Rewrite the fraction and use the product rule. $f(x) = x^{-2}\cos x$

$$f'(x) = \cos x\left(-2x^{-3}\right) - x^{-2}(-\sin x) = x^{-3}(-2\cos x - x\sin x) = \boxed{\frac{-2\cos x - x\sin x}{x^3}}$$

Ex2: Differentiate $y = \dfrac{\sin x}{\cos x}$.

$$y' = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$$

Note: The example above shows how to differentiate $y = \dfrac{\sin x}{\cos x} = \tan x$.

Teacher Note: You can have students rewrite all of the other trig functions in terms of sine and cosine to derive the following derivative rules.

Derivatives of Trigonometric Functions:

$$\boxed{\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x\tan x \qquad \frac{d}{dx}\csc x = -\csc x\cot x \qquad \frac{d}{dx}\cot x = -\csc^2 x}$$

Memory Aid – Note that all of the cofunctions’ derivatives are negative.

Ex3: Differentiate $y = \dfrac{4}{\cos x}$.

To avoid using the quotient rule, we can rewrite the function as $y = 4\sec x$. $\boxed{y' = 4\sec x\tan x}$

Ex4: Write the equation of the tangent line of the function $y = \sin x + \cos x$ at $x = \pi$.

Find the slope of the tangent line: $m = y' = \cos x - \sin x \;\Rightarrow\; y'(\pi) = \cos\pi - \sin\pi = -1$

Find the ordered pair on the function: $y(\pi) = \sin\pi + \cos\pi = -1 \;\Rightarrow\; (\pi, -1)$

Write the equation of the line: $\boxed{y + 1 = -1(x - \pi)} \Rightarrow \boxed{y = -x + \pi - 1}$

Teacher Note: Many students will need to review the trigonometric functions, especially the unit circle. Be sure they can evaluate trig functions quickly, in radians. Suggestion – give timed quizzes on the unit circle.

Simple Harmonic Motion: the motion of a weight bobbing up and down on the end of a spring

Ex5: A weight hanging from a spring is stretched 8 inches beyond its rest position $(s = 0)$ and released at time $t = 0$ to bob up and down. Its position at any later time *t* is $s = 8\cos t$. What are its velocity and acceleration at time *t*?

Velocity: $v(t) = s'(t) = -8\sin t$

Acceleration: $a(t) = v'(t) = -8\cos t$

Jerk: rate of change of acceleration (third derivative of the position function)

Jerk = $a'(t) = s'''(t)$

You Try: Use a trigonometric identity to differentiate $f(x) = \cos 2x$. (Hint: Use $\cos 2x = \cos x \cdot \cos x - \sin x \cdot \sin x$.) Express the derivative in terms of $\sin 2x$.

QOD: Calculate higher order derivatives for $y = \sin x$. Then use any patterns you discover to find $y = \sin^{(215)} x$.

Syllabus Objective: 2.4 – The student will use the chain rule to differentiate the composition of functions.

Derivatives of Composite Functions: The Chain Rule

Recall: $(f \circ g)(x) = f(g(x))$; Read "*f* of *g* of *x*."

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) \qquad \text{or} \qquad y = f(u) \quad u = g(x), \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Take the derivative of the "outside" function first, then multiply by the derivative of the "inside" function.

Ex1: Find the derivative of $y = \sin\left(x^2 + x\right)$.

Let $u = x^2 + x$. So $y = \sin u$. Differentiate *y* with respect to *u*: $\frac{dy}{du} = \cos u$

Now differentiate *u* with respect to *x*: $\frac{du}{dx} = 2x + 1$

So $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \Rightarrow \frac{dy}{dx} = \cos u \cdot (2x+1) = \boxed{(2x+1)\cos\left(x^2 + x\right)}$

Ex2: Differentiate $f(x) = \frac{1}{(2x-3)^3}$.

Rewrite as a negative exponent (avoid the quotient rule!): $f(x) = (2x-3)^{-3}$

Instead of using *u*-substitution, we will differentiate the **outside** function using the power rule and then **multiply** by the derivative of the **inside** function, which is the base of the power.

$$f'(x) = -3(2x-3)^{-4}(2) = \boxed{\frac{-6}{(2x-3)^4}}$$

Teacher Note: Allow students to use the *u*-substitution method until they are comfortable with identifying the outside and inside functions.

More Links in the Chain

Ex3: Find $\frac{dy}{dx}$ if $y=\tan(3-\cos 7x)$.

The outside function is $y=\tan u$. So $\frac{dy}{dx}=\sec^2(3-\cos 7x)\cdot\frac{d}{dx}(3-\cos 7x)$

Now differentiate $u=3-\cos 7x$: $\frac{dy}{dx}=\sec^2(3-\cos 7x)\cdot(\sin 7x)\frac{d}{dx}(7x)$

Differentiate the final "link" in the chain: $\frac{dy}{dx}=\sec^2(3-\cos 7x)\cdot(\sin 7x)\cdot 7=\boxed{7\sin(7x)\sec^2(3-\cos 7x)}$

Using the Chain Rule with a Table of Values

Ex4: Evaluate the derivatives using the table below.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	2	3	1/3	−3
3	3	−4	2π	5

a) $f(g(x))$ at $x=2$

Differentiate using the chain rule: $f'(g(x))\cdot g'(x)$

At $x=2$, using the table, we have $f'(g(2))\cdot g'(2)=f'(3)\cdot(-3)=2\pi\cdot(-3)=\boxed{-6\pi}$

b) $g(f(x))$ at $x=2$

Differentiate using the chain rule: $g'(f(x))\cdot f'(x)$

At $x=2$, using the table, we have $g'(f(2))\cdot f'(2)=g'(2)\cdot\left(\frac{1}{3}\right)=(-3)\cdot\left(\frac{1}{3}\right)=\boxed{-1}$

c) $\frac{1}{g^2(x)}$ at $x=3$

Rewrite as a negative exponent: $g^{-2}(x)$ Differentiate using the chain rule: $-2g^{-3}(x)\cdot g'(x)$

At $x=3$, using the table, we have $-2g^{-3}(3)\cdot g'(3)=-2\cdot\frac{1}{(-4)^3}\cdot 5=\frac{-10}{-64}=\boxed{\frac{5}{32}}$

d) $\sqrt{f(x)}$ at $x = 2$

Rewrite the radical as an exponent: $(f(x))^{\frac{1}{2}}$ Differentiate using the chain rule: $\frac{1}{2}(f(x))^{-\frac{1}{2}} \cdot f'(x)$

At $x = 2$, using the table, we have $\frac{1}{2}(f(2))^{-\frac{1}{2}} \cdot f'(2) = \frac{1}{2}(2)^{-\frac{1}{2}} \cdot \frac{1}{3} = \boxed{\frac{1}{6\sqrt{2}}}$

<u>You Try:</u> Find $(f \circ g)'$ at $x = -1$ if $f(u) = 1 - \frac{1}{u}$ and $u = g(x) = \frac{1}{1-x}$.

<u>QOD:</u> Explain the significance of the name of the "Chain" Rule.

<u>Sample AP Calculus AB Exam Question(s) (taken from the released 2003 MC AP Exam):</u>

1. If $y = (x^3 + 1)^2$, then $\frac{dy}{dx} =$

   (A) $(3x^2)^2$

   (B) $2(x^3 + 1)$

   (C) $2(3x^2 + 1)$

   (D) $3x^2(x^3 + 1)$

   (E) $6x^2(x^3 + 1)$

2. If $y = x^2 \sin 2x$, then $\frac{dy}{dx} =$

   (A) $2x \cos 2x$

   (B) $4x \cos 2x$

   (C) $2x(\sin 2x + \cos 2x)$

   (D) $2x(\sin 2x - x \cos 2x)$

   (E) $2x(\sin 2x + x \cos 2x)$

3. Let f be a differentiable function with $f(2)=3$ and $f'(2)=-5$, and let g be the function defined by $g(x)=x\,f(x)$. Which of the following is an equation of the line tangent to the graph of g at the point where $x=2$?

   (A) $y=3x$

   (B) $y-3=-5(x-2)$

   (C) $y-6=-5(x-2)$

   (D) $y-6=-7(x-2)$

   (E) $y-6=-10(x-2)$

Syllabus Objectives: 2.5 – The student will differentiate an implicitly-defined function.

Implicitly-Defined Function: a function with multiple variables that is not solved for one of the variables

For example: $4(x-1)^2 + y^2 = 25$

Note: An explicitly-defined function is a function that is written in function form, $y = f(x)$.

Implicit Differentiation: differentiating a function that is not written as an explicit formula

Use the following steps:

I. Differentiate both sides of the equation with respect to *x* using the chain rule.

II. Collect all terms with $\frac{dy}{dx}$ on one side of the equation.

III. Factor out $\frac{dy}{dx}$.

IV. Solve for $\frac{dy}{dx}$.

Ex1: Find $\frac{dy}{dx}$ for the function $x^2 - xy + y^2 = 7$.

I. $2x - \left(y \cdot 1 + x \cdot \frac{dy}{dx} \right) + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow 2x - y - x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = 0$

Note: When differentiating with respect to *x*, the derivative of *x* is $\frac{dx}{dx} = 1$, and the derivative of *y* is $\frac{dy}{dx}$.

II. $-x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = -2x + y$

III. $\frac{dy}{dx}(-x + 2y) = -2x + y$

IV. $\frac{dy}{dx} = \boxed{\frac{-2x + y}{-x + 2y}}$

Note: There are many ways of writing the correct answer, for example, $\frac{dy}{dx} = \boxed{\frac{y - 2x}{2y - x}} = \boxed{\frac{2x - y}{x - 2y}}$. Watch for these on multiple choice selections.

Ex2: Find the tangent and normal lines of $2xy + \pi \sin y = 2\pi$ at the point $\left(1, \frac{\pi}{2}\right)$.

I. $y \cdot 2 + 2x \cdot \frac{dy}{dx} + \pi \cos y \cdot \frac{dy}{dx} = 0$

II. $2x \cdot \frac{dy}{dx} + \pi \cos y \cdot \frac{dy}{dx} = -2y$

III. $\frac{dy}{dx}\left(2x + \pi \cos y\right) = -2y$

IV. $\frac{dy}{dx} = \frac{-2y}{2x + \pi \cos y}$

Evaluate $\frac{dy}{dx}$ at $\left(1, \frac{\pi}{2}\right)$ to find the slope of the tangent line. $\frac{dy}{dx} = \frac{-2\frac{\pi}{2}}{2 \cdot 1 + \pi \cos\left(\frac{\pi}{2}\right)} = \frac{-\pi}{2 + \pi \cdot 0} = -\frac{\pi}{2}$

Equation of tangent line: $\boxed{y - \frac{\pi}{2} = -\frac{\pi}{2}(x-1)}$ Equation of normal line: $\boxed{y - \frac{\pi}{2} = \frac{2}{\pi}(x-1)}$

Finding a Second Derivative

Ex3: Find $\frac{d^2y}{dx^2}$ for the function $4y^3 = 9 - 5x^2$.

$12y^2 \cdot \frac{dy}{dx} = -10x$ $\frac{dy}{dx} = \frac{-10x}{12y^2} = -\frac{5x}{6y^2}$

$\frac{dy}{dx} = -\frac{5x}{6y^2}$ Differentiating again, we have: $\frac{d^2y}{dx^2} = -\frac{6y^2 \cdot 5 - 5x \cdot 12y \cdot \frac{dy}{dx}}{\left(6y^2\right)^2}$

Now substitute $\frac{dy}{dx} = -\frac{5x}{6y^2}$ and simplify: $\frac{d^2y}{dx^2} = -\frac{6y^2 \cdot 5 - 5x \cdot 12y \cdot -\frac{5x}{6y^2}}{\left(6y^2\right)^2} = -\frac{30y^2 - \frac{10x^2}{y}}{36y^4}$

Multiply every term by *y* to eliminate the complex fraction: $\frac{d^2y}{dx^2} = -\frac{30y^3 - 10x^2}{36y^5}$ or $\boxed{\frac{d^2y}{dx^2} = \frac{5x^2 - 15y^3}{18y^5}}$

You Try: Write the equation of the tangent and normal lines of the curve $x^2 - \sqrt{3}xy + 2y^2 = 5$ at $x = \sqrt{3}$.

QOD: When is it appropriate to use implicit differentiation to calculate a derivative?

Sample AP Calculus AB Exam Question(s):

What is the slope of the line tangent to the curve $3y^2 - 2x^2 = 6 - 2xy$ at the point $(3,2)$?

(A) 0

(B) $\frac{4}{9}$

(C) $\frac{7}{9}$

(D) $\frac{6}{7}$

(E) $\frac{5}{3}$

Syllabus Objective: 2.6 – The student will differentiate the inverse of an elementary function.

Derivatives of Inverse Functions: the derivative at $(a, f(a))$ of $f(x)$ is the *reciprocal* of the derivative at $(f(a), a)$ of the *inverse*, $f^{-1}(x)$, of $f(x)$

Note: Because the inverse of a function is the reflection over the line $y = x$, a change in *y* becomes a change in *x*, and a change in *x* becomes a change in *y*. Thus $\frac{dy}{dx}$ becomes $\frac{dx}{dy}$, which is why we use the reciprocal.

Ex1: Let $f(x) = x^3 - 6x^2 + 8$. Find the derivative of $f^{-1}(x)$ at the point $(3,1)$.

Since the point $(3,1)$ is on the graph of $f^{-1}(x)$, we know that the point $(1,3)$ is on the graph of $f(x)$.

Find the derivative of $f(x)$ at $(1,3)$: $f'(x) = 3x^2 - 12x \Rightarrow f'(1) = -9$

The derivative of $f^{-1}(x)$ at the point $(3,1)$ is the reciprocal: $\boxed{-\frac{1}{9}}$

Derivatives of Inverse Trigonometric Functions

Derivative of Arcsine

$y = \sin^{-1} x$ can be rewritten as $\sin y = x$, and is differentiable on the open interval $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Using implicit differentiation, we have $\cos y \frac{dy}{dx} = 1$, or $\frac{dy}{dx} = \frac{1}{\cos y}$.

We need the derivative in terms of *x*, so we will use a Pythagorean identity to replace $\cos y$.

$$\sin^2 y + \cos^2 y = 1 \Rightarrow \cos y = \pm\sqrt{1 - \sin^2 y}$$

Note: We need only use the positive square root because $\cos y$ is positive on the interval $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Substituting, we have $\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}}$, and since $\sin y = x$, we have $\boxed{\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1 - x^2}}}$

Derivative of Arctangent

$y=\tan^{-1}x$ can be rewritten as $\tan y = x$, and is differentiable for all real numbers.

Using implicit differentiation, we have $\sec^2 y\cdot\frac{dy}{dx}=1$ or $\frac{dy}{dx}=\frac{1}{\sec^2 y}$.

We need the derivative in terms of *x*, so we will use a Pythagorean identity to replace $\cos y$.

$$\tan^2 y+1=\sec^2 y$$

Substituting, we have $\frac{dy}{dx}=\frac{1}{1+\tan^2 y}$, and since $\tan y = x$, we have $\boxed{\frac{d}{dx}\left[\tan^{-1}x\right]=\frac{1}{1+x^2}}$

<u>On Your Own</u>: Using a method similar to the ones shown above, find the derivative of arcsecant.

Solution: $\boxed{\frac{d}{dx}\left[\sec^{-1}x\right]=\frac{1}{|x|\sqrt{x^2-1}}}$

<u>Teacher Note:</u> It will be much easier for students to memorize these once they have derived them using the Pythagorean identities. It is **not** recommended to just give them the formulas!

Cofunctions of the Inverse Trigonometric Functions

Recall: $\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x$ $\quad\cot^{-1}x=\frac{\pi}{2}-\tan^{-1}x$ $\quad\csc^{-1}x=\frac{\pi}{2}-\sec^{-1}x$

Derivatives of the Cofunctions of Inverse Trigonometric Functions (differentiate the equations shown above)

$\frac{d}{dx}\left[\cos^{-1}x\right]=-\frac{1}{\sqrt{1-x^2}}$ $\quad\frac{d}{dx}\left[\cot^{-1}x\right]=-\frac{1}{1+x^2}$ $\quad\frac{d}{dx}\left[\csc^{-1}x\right]=-\frac{1}{|x|\sqrt{x^2-1}}$

So, the derivatives of the cofunctions are the *opposite* of the derivatives of their respective inverse trig functions.

Ex2: Find $\frac{dy}{dx}$ for the function $y=\cos^{-1}\left(x^2\right)$.

We must use the chain rule with $u=x^2$. $\quad\frac{dy}{du}=-\frac{1}{\sqrt{1-u^2}}$

So $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=-\frac{1}{\sqrt{1-u^2}}\cdot(2x)=-\frac{2x}{\sqrt{1-\left(x^2\right)^2}}=\boxed{-\frac{2x}{\sqrt{1-x^4}}}$

Ex3: Find an equation of the tangent line to the curve $y = \tan^{-1}\sqrt{x}$ at $x = 1$.

Differentiate and find the slope of the tangent line: $y' = \frac{1}{2}x^{-\frac{1}{2}} \cdot \frac{1}{\left(\left(\sqrt{x}\right)^2 + 1\right)} = \frac{1}{2\sqrt{x}\left(x+1\right)} \Rightarrow y'(1) = \frac{1}{4}$

At $x = 1$, $y = \tan^{-1}\sqrt{1} = \frac{\pi}{4}$. So an equation for the tangent line is $\boxed{y - \frac{\pi}{4} = \frac{1}{4}(x-1)}$

<u>You Try:</u> Differentiate $y = \sin^{-1}\sqrt{2t}$.

<u>QOD:</u> What is the relationship between the derivative of a function and the derivative of its inverse?

<u>Sample AP Calculus AB Exam Question(s) (taken from the released 2003 MC AP Exam):</u>

Let f be the function defined by $f(x) = x^3 + x$. If $g(x) = f^{-1}(x)$ and $g(2) = 1$, what is the value of $g'(2)$?

(A) $\frac{1}{13}$

(B) $\frac{1}{4}$

(C) $\frac{7}{4}$

(D) 4

(E) 13

Syllabus Objective: 2.7 – The student will evaluate the derivative of a function by the technique of logarithmic differentiation.

Derivative of $y = e^x$

Recall: $\lim_{h\to 0}\frac{e^h - 1}{h} = 1$ (This can be shown graphically or by using a table of values.)

$$\frac{d}{dx}\left[e^x\right] = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x\left(e^h-1\right)}{h} = \lim_{h\to 0} e^x \cdot \frac{e^h-1}{h} = \lim_{h\to 0} e^x \cdot 1 = e^x \qquad \text{So, } \boxed{\frac{d}{dx}\left[e^x\right] = e^x}$$

Ex: Find y' if $y = 3e^{5x^2}$.

Use the chain rule: $y' = 3\cdot e^{5x^2}\cdot 10x = \boxed{30x\cdot e^{5x^2}}$

Recall:
$e^{\ln a} = a$

Derivative of $y = a^x$, $a > 0$, $a \neq 1$

Note that $a^x = e^{x\ln a}$, as shown here: $e^{x\ln a} = e^{\ln a^x} = a^x$

$$\frac{d}{dx}\left[a^x\right] = \frac{d}{dx}\left[e^{x\ln a}\right] = e^{x\ln a}\cdot \ln a = a^x \cdot \ln a \qquad \text{So, } \boxed{\frac{d}{dx}\left[a^x\right] = a^x \cdot \ln a}$$

Ex: At what point on the graph of $y = 2^t - 3$ does the tangent line have slope 21?

Slope of the tangent line: $y' = 2^t \cdot \ln 2 \qquad 21 = 2^t\cdot \ln 2 \Rightarrow 2^t = \frac{21}{\ln 2}$

Solve for t by taking the ln of both sides. $\ln 2^t = \ln\left(\frac{21}{\ln 2}\right) \Rightarrow t(\ln 2) = \ln\left(\frac{21}{\ln 2}\right) \Rightarrow t = \frac{\ln\left(\frac{21}{\ln 2}\right)}{\ln 2} \approx 4.921$

$y = 2^t - 3 \Rightarrow y = 2^{\frac{\ln\left(\frac{21}{\ln 2}\right)}{\ln 2}} - 3 \approx 27.295$ Solution: $\boxed{(4.921, 27.295)}$

Derivative of $y = \ln x$

$y = \ln x$ can be rewritten in exponential form as $e^y = x$.

Using implicit differentiation, we have $e^y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{e^y}$. Since $e^y = x$, $\boxed{\frac{d}{dx}\left[\ln x\right] = \frac{1}{x}}$.

Ex: A line with slope *m* passes through the origin and is tangent to $y = \ln x$. What is *m*?

Differentiate: $\dfrac{dy}{dx} = \dfrac{1}{x}$. Using the point on $y = \ln x$, $(a, \ln a)$, we have $\left.\dfrac{dy}{dx}\right|_{x=a} = \dfrac{1}{a}$

Since the line passes through the origin, $(0,0)$, and the point $(a, \ln a)$, its slope is $m = \dfrac{\ln a - 0}{a - 0} = \dfrac{\ln a}{a}$.

Thus, $\dfrac{1}{a} = \dfrac{\ln a}{a} \Rightarrow \ln a = 1 \Rightarrow a = e$. So $\boxed{m = \dfrac{1}{e}}$.

Derivative of $y = \log_a x$

Rewrite using the change of base formula: $y = \log_a x = \dfrac{\ln x}{\ln a} = \dfrac{1}{\ln a} \cdot \ln x$

$\dfrac{d}{dx}[\log_a x] = \dfrac{d}{dx}\left[\dfrac{1}{\ln a} \cdot \ln x\right] = \dfrac{1}{\ln a} \cdot \dfrac{1}{x}$ So, $\boxed{\dfrac{d}{dx}[\log_a x] = \dfrac{1}{x \ln a}}$

Ex: Find $\dfrac{dy}{dx}$ for $y = \log_a a^{\sin x}$.

$\dfrac{dy}{dx} = \dfrac{1}{a^{\sin x} \ln a} \cdot a^{\sin x} \ln a \cdot \cos x = \boxed{\cos x}$ Note: $y = \log_a a^{\sin x} = \sin x$!

Logarithmic Differentiation: used when a function has a variable raised to a variable power

Ex: Find $\dfrac{dy}{dx}$ for $y = x^x$, $x > 0$.

Note: We **cannot** use the power rule, because the exponent is not a constant. Also, we **cannot** use the formula for differentiating $y = a^x$, because the base is not a constant. So we must use *logarithmic differentiation*.

Take the natural log of both sides: $\ln y = \ln x^x \Rightarrow \ln y = x \ln x$

Use implicit differentiation: $\dfrac{1}{y}\dfrac{dy}{dx} = \ln x \cdot 1 + x \cdot \dfrac{1}{x} \Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = \ln x + 1 \Rightarrow \dfrac{dy}{dx} = y(\ln x + 1)$

Substitute $y = x^x$: $\boxed{\dfrac{dy}{dx} = x^x(\ln x + 1)}$

You Try: Differentiate $y = x^{\tan x}$.

QOD: Prove that for any constant k, the derivative of $\ln(kx)$ is $\frac{1}{x}$ in two ways: 1) Using the chain rule, and 2) Using the properties of logarithms.

Sample AP Calculus AB Exam Question(s) (taken from the released 2003 MC AP Exam):

If $f(x) = \ln\left(x + 4 + e^{-3x}\right)$, then $f'(0)$ is

(A) $-\frac{2}{5}$

(B) $\frac{1}{5}$

(C) $\frac{1}{4}$

(D) $\frac{2}{5}$

(E) nonexistent