AP Calculus Notes: Unit 2 – Derivatives (Part One)

Syllabus Objectives: 2.1 – The student will explore the definition of the derivative and its alternate forms. 2.2 – The student will find the derivative of an elementary function using the definition of the derivative. 2.11 – The student will sketch curves using derivatives and limits.

Definition:

The DERIVATIVE of a function *f* with respect to the variable *x* is the function f' whose value at *x* is

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad \text{(provided the limit exists)}$$

Derivative Notation:

y' "y prime"

$\frac{dy}{dx}$ "dy dx" or the derivative of y with respect to x"

$\frac{df}{dx}$ "the derivative of f with respect to x"

$\frac{d}{dx}f(x)$ "d dx of f at x" or "the derivative of f at x"

Ex1: Differentiate $y = 2x^2$.

$$y' = \lim_{h\to 0}\frac{2(x+h)^2-2x^2}{h} = \lim_{h\to 0}\frac{2x^2+4xh+h^2-2x^2}{h} = \lim_{h\to 0}\frac{4xh+h^2}{h} = \lim_{h\to 0}4x+h = 4x+0 \Rightarrow \boxed{y'=4x}$$

Ex2: Find the derivative of $f(x) = x^3$

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^3-x^3}{h} = \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} = \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0}3x^2+3xh+h^2 \Rightarrow \boxed{f'(x)=3x^2}$$

Ex3: Find $f'(x)$ for the function $f(x)=\sqrt{x+1}$

$f'(x) = \lim_{h\to 0}\frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}$ Multiply numerator and denominator by the conjugate of the numerator:

$$= \lim_{h\to 0}\frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}\cdot\frac{\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}{\left(\sqrt{x+h+1}+\sqrt{x+1}\right)} = \lim_{h\to 0}\frac{x+h+1-(x+1)}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)} = \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}$$

$$= \lim_{h\to 0}\frac{1}{\left(\sqrt{x+h+1}+\sqrt{x+1}\right)} = \frac{1}{\left(\sqrt{x+1}+\sqrt{x+1}\right)} \Rightarrow \boxed{f'(x)=\frac{1}{2\sqrt{x+1}}}$$

Alternate Definition of a Derivative:

The DERIVATIVE of the function *f* at the point $x = a$ is the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$, provided the limit exists.

Ex: Use the alternate definition to differentiate $f(x) = x^2 - 2$.

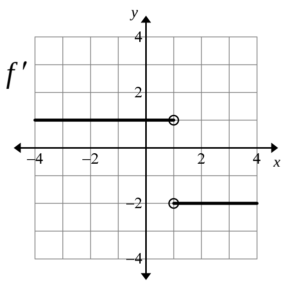
$$f'(a) = \lim_{x \to a} \frac{x^2 - 2 - (a^2 - 2)}{x - a} = \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} \frac{(x+a)(x-a)}{x - a} = \lim_{x \to a} x + a = a + a = 2a$$

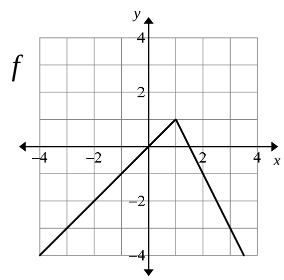
The derivative at a point *a* is 2*a*, so $\boxed{f'(x) = 2x}$.

Note: The derivative of a point in graphical terms is a *slope.*

Graphing f from f'

Ex: Sketch the graph of a continuous function f using the graph of f', if $f(0) = 0$.





Solution: To the left of 1, *f* has a derivative (slope) of 1. To the right of 1, *f* has a derivative (slope) of −2. Also, *f* passes through the origin.

One-sided Derivatives:

Right-hand derivative at *a*:

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

Left-hand derivative at *a*:

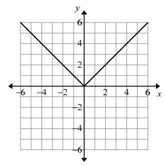
$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

Note: If the right- and left-hand derivatives are not equal at an interior point *a*, then the function does not have a derivative at *a*.

Special Case – Endpoints: A left endpoint has a derivative if the right-hand derivative exists, and a right endpoint has a derivative if the left-hand derivative exists.

Ex: Find the one-sided derivatives at $x = 0$ for the function $y = |x|$.

Use the graph of $y = |x|$.



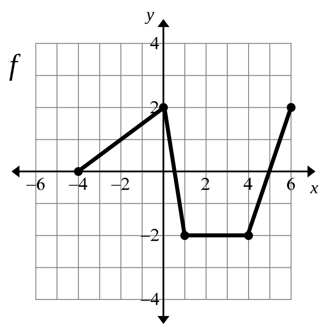
Remember – A derivative is SLOPE.

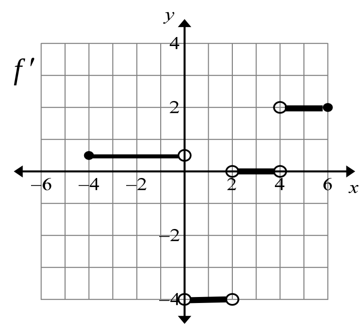
Right-Hand Derivative: the derivative (slope) of the function to the right of $x = 0$ is $\boxed{1}$.

Left-Hand Derivative: the derivative (slope) of the function to the left of $x = 0$ is $\boxed{-1}$.

Note: Because the right-hand and left-hand derivatives are not equal at $x = 0$, the function is not differentiable at $x = 0$.

Ex: Graph the derivative of the function shown.





Solution: Use the slopes of the function on the following intervals to graph f':

$[-4,0)$, slope $= \frac{1}{2}$ $\quad$ $(0,2)$, slope $= -4$ $\quad$ $(2,4)$, slope $= 0$ $\quad$ $(4,6]$, slope $= 2$

<u>You Try:</u> Evaluate the following for the function. $f(x) = \begin{cases} x^2, & x \le 1 \\ 2x, & x > 1 \end{cases}$

a) $f'(x), x < 1$

b) $f'(x), x < 1$

c) $\lim_{x\to 1^-} f'(x)$

d) $\lim_{x\to 1^+} f'(x)$

e) Does $\lim_{x\to 1} f'(x)$ exist?

f) Find the right-hand derivative at $x = 1$.

g) Find the left-hand derivative at $x = 1$.

h) Does $f'(1)$ exist?

<u>QOD:</u> A race car driver states that his "derivative" was 120 at a given time. What does the derivative represent? What could the units be for the derivative?

Sample AP Calculus AB Exam Question(s) (taken from the released 2003 MC AP Exam):

$$f(x) = \begin{cases} x+2 & \text{if } x \le 3 \\ 4x-7 & \text{if } x > 3 \end{cases}$$

Let f be the function given above. Which of the following statements are true about f?

I. $\lim_{x\to 3} f(x)$ exists

II. f is continuous at $x = 3$.

III. f is differentiable at $x = 3$.

(A) None
(B) I only
(C) II only
(D) I and II only
(E) I, II, and III

Syllabus Objectives: 2.1 – The student will explore the definition of the derivative and its alternate forms. 2.2 – The student will find the derivative of an elementary function using the definition of the derivative.

Differentiability

When $f'(a)$ does not exist:

a. **Corner** – one-sided derivatives differ

Ex1: $y=|x|$ at $x=0$

The left-hand derivative at $x=0$ is -1, and the right-hand derivative at $x=0$ is 1.

b. **Cusp** – the slopes of the secant lines approach ∞ from one side and $-\infty$ from the other

Ex2: $y=x^{\frac{2}{3}}$ at $x=0$

The slopes of the secant lines approach $-\infty$ from the left of $x=0$, and they approach ∞ from the right of $x=0$.

c. **Vertical Tangent** – the slopes of the secant lines approach either ∞ or $-\infty$ from both sides

Ex3: $y=\sqrt[3]{x}$ at $x=0$

The slopes of the secant lines approach ∞ from both sides.

d. **Discontinuity** – the function has a discontinuity

Ex4: $f(x)=\begin{cases} 1 & x\le 0 \\ -1 & x>0 \end{cases}$ at $x=0$

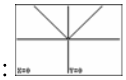
The function has a jump discontinuity at $x=0$.

Note: Differentiability implies continuity. (The converse is NOT true.)

Teacher Note: Have students graph the functions above to get a visual of each type.

<u>Local Linearity</u>: a function differentiable at a point resembles the tangent line at that point as you get close to the point

Ex5: Graph the functions $f(x) = |x|$ and $g(x) = \sqrt{x^2 + 0.001}$. Zoom in to see if either function is differentiable at $x = 0$.

Graph of $f(x) = |x|$ after zooming in several times on $x = 0$: 

Graph of $g(x) = \sqrt{x^2 + 0.001}$ after zooming in several times on $x = 0$: 

Note that the graph of $f(x) = |x|$ looks the same after zooming in several times, whereas the graph of $g(x) = \sqrt{x^2 + 0.001}$ appears to straighten out. Because of this local linearity, we know that only $g(x) = \sqrt{x^2 + 0.001}$ is differentiable at $x = 0$.

<u>Calculating Numerical Derivatives on the Graphing Calculator</u>

Ex6: Calculate the derivative of $f(x) = x^2 - 2$ at $x = 3$ on the graphing calculator.

Use the nDeriv command (located in the MATH menu, #8). To use this feature, type in the function, variable of differentiation, and numerical value.

```
nDeriv(X²-2,X,3)
                   6
█
```

Notation: $\boxed{f'(3) = 6}$ or $\boxed{\frac{d}{dx} f(3) = 6}$

Note: To check our answer, we can use a previous example from this unit, where we found that $f'(x) = 2x$, so $f'(3) = 2(3) = 6$.

Caution: Because the TI-83/84 (or any calculator without a Computer Algebra System) uses a difference quotient to find numerical derivatives, it may give a false value for a derivative that does not exist.

<u>Graphing a Derivative on the Graphing Calculator</u>

Ex7: Let $f(x) = \sin x$. Use nDeriv to graph $f'(x)$ on the same viewing window. Can you guess what function $f'(x)$ is?

Type in the following for Y1 and Y2.

Note that the graph of $f'(x)$ is bold.

```
  Plot1  Plot2  Plot3
\Y1=sin(X)
\Y2=nDeriv(Y1,X,
X)
\Y3=
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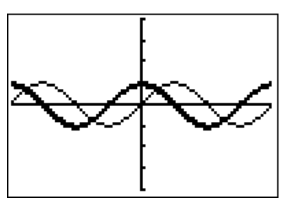


The graph of $f'(x)$ is $\cos x$.

<u>Theorem:</u> If *f* has a derivative at $x = a$, then *f* is continuous at $x = a$. So, differentiability implies continuity.

Proof: Show that $\lim_{x \to a} f(x) = f(a)$ or $\lim_{x \to a} f(x) - f(a) = 0$.

$$\lim_{x \to a}\left[f(x) - f(a)\right] = \lim_{x \to a}\left[(x-a)\frac{f(x)-f(a)}{(x-a)}\right] = \lim_{x \to a}(x-a)\cdot\lim_{x \to a}\frac{f(x)-f(a)}{(x-a)} = 0\cdot\lim_{x \to a}\frac{f(x)-f(a)}{(x-a)} = 0$$

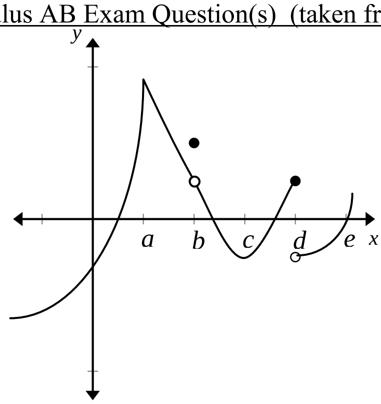
<u>Intermediate (Mean) Value Theorem:</u> If *f* is differentiable on a closed interval $[a,b]$, then f' takes on every value between $f'(a)$ and $f'(b)$.

Think About It: Suppose you drive from your house to the highway. You begin with the car parked (0 mph) and end up cruising at 65 mph on the highway. At some point on your trip were you ever driving at 2 mph? How about 55 mph? How does this scenario relate to the intermediate value theorem?

<u>You Try:</u> Find all values of *x* for which the function $f(x) = \sqrt[3]{x-2} + 5$ is differentiable. For any values of *x* for which the function fails to be differentiable, identify why.

<u>QOD:</u> Differentiability implies continuity, as shown in the theorem above. Is the converse to this theorem true? Explain why or why not.

<u>Sample AP Calculus AB Exam Question(s) (taken from the released 2003 MC AP Exam):</u>



Graph of *f*

1. The graph of a function *f* is shown above. At which value of *x* is *f* continuous, but not differentiable?

   (A) *a*

   (B) *b*

   (C) *c*

   (D) *d*

   (E) *e*

2. The function f is continuous for $-2 \le x \le 1$ and differentiable for $-2 < x < 1$. If $f(-2) = -5$ and $f(1) = 4$, which of the following statements could be false?

   (A) There exists *c*, where $-2 < x < 1$, such that $f(c) = 0$.

   (B) There exists *c*, where $-2 < x < 1$, such that $f'(c) = 0$.

   (C) There exists *c*, where $-2 < x < 1$, such that $f(c) = 3$.

   (D) There exists *c*, where $-2 < x < 1$, such that $f'(c) = 3$.

   (E) There exists *c*, where $-2 \le x \le 1$, such that $f(c) \ge f(x)$ for all *x* on the closed interval $-2 \le x \le 1$.

Syllabus Objectives: 2.3 – The student will differentiate the sum, product, and quotient of elementary functions. 2.8 – The student will find higher-order derivatives of elementary functions. 2.10 – The student will find the equation of the tangent line and/or the normal to a given curve at a given point.

Rules of Differentiation:

Teacher Note: Have students hypothesize the rules based upon the numerous examples they have completed using the derivative definition.

1. Derivative of a Constant Function: $\frac{d}{dx}(c)=0$

   Ex1: Differentiate $f(x)=3$. $\boxed{f'(x)=0}$

2. Power Rule: $\frac{d}{dx}(x^n)=nx^{n-1}$

   Ex2: Differentiate $f(x)=x^3$. $\boxed{f'(x)=3x^2}$

3. Constant Multiple Rule: $\frac{d}{dx}(cu)=c\frac{du}{dx}$

   Ex3: Differentiate $f(x)=-4x^2$. $\boxed{f'(x)=-4(2x)=-8x}$

4. Sum and Difference Rule: $\frac{d}{dx}(u\pm v)=\frac{d}{dx}(u)\pm\frac{d}{dx}(v)$

   Ex4: Differentiate $f(x)=3x^5-2x+5$. $\boxed{f'(x)=15x^4-2}$

5. Product Rule: $\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}$

Caution: The derivative of a product is NOT the product of the derivatives! Case in point:

$$\frac{d}{dx}(x\cdot x)=\frac{d}{dx}(x^2)=2x \text{ and } \frac{d}{dx}(x)\cdot\frac{d}{dx}(x)=1\cdot 1=1$$

Ex5: Find y' if $y=(x^2+3)(x^4+x)$.

Let $u=x^2+3$ and $v=x^4+x$, and apply the product rule:

$$y'=(x^4+x)(2x)+(x^2+3)(4x^3+1)=2x^5+2x^2+4x^5+12x^3+x^2+3=\boxed{6x^5+12x^3+3x^2+3}$$

On Your Own: Algebraically simplify the function in the example above, then differentiate. You should obtain the same derivative as we did above.

6. Quotient Rule: $$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Caution: The derivative of a quotient is NOT the quotient of the derivatives!

Memory Aid for Quotient Rule: low d-high minus high d-low; square the bottom and away we go!
("low" = denominator; "high" = numerator)

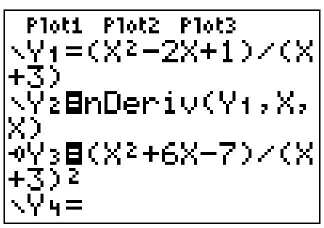
Ex6: Differentiate $f(x)=\dfrac{x^2-2x+1}{x+3}$.

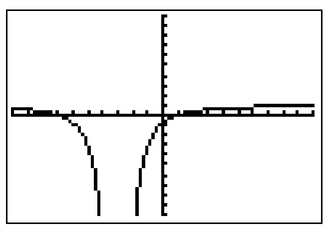
Let $u=x^2-2x+1$ and $v=x+3$; then apply the quotient rule.

$$f'(x)=\frac{(x+3)(2x-2)-(x^2-2x+1)(1)}{(x+3)^2}=\frac{2x^2+4x-6-x^2+2x-1}{(x+3)^2}=\boxed{\frac{x^2+6x-7}{(x+3)^2}}$$

<u>Checking Derivatives on the Graphing Calculator</u>

To check our answer to the example above, we can graph the function we came up with as the derivative, and also have the calculator graph the derivative of $f(x)$ using nDeriv. If the graphs match, we are correct!





Note: Turn off the graph of the actual function (Y1), and change the format of Y3 to "bubble" so we can see it graph.

<u>Algebraic Manipulation</u>

Ex7: Calculate the derivative of $y=\dfrac{x^2+5x-1}{2x^2}$ at $x=-1$.

We could use the quotient rule to calculate the derivative, but it would make it easier if we simplified first.

$$y=\frac{x^2+5x-1}{2x^2}=\frac{x^2}{2x^2}+\frac{5x}{2x^2}-\frac{1}{2x^2}=\frac{1}{2}+\frac{5}{2x}-\frac{1}{2x^2}=\frac{1}{2}+\frac{5}{2}x^{-1}-\frac{1}{2}x^{-2}$$

Now we can use the power rule: $\dfrac{dy}{dx}=-\dfrac{5}{2}x^{-2}+x^{-3}$ So, $\left.\dfrac{dy}{dx}\right|_{x=-1}=-\dfrac{5}{2}(-1)^{-2}+(-1)^{-3}=-\dfrac{5}{2}-1=\boxed{-\dfrac{7}{2}}$

Ex8: Use the given information to evaluate the following derivatives at $x = 2$.
$u(2) = 4,\ u'(2) = -3,\ v(2) = 1,\ v'(2) = 5$

Note: We will differentiate FIRST; then substitute in numerical values.

a. $\dfrac{d}{dx}(uv)$

$$\frac{d}{dx}(uv) = v\frac{d}{dx}(u) + u\frac{d}{dx}(v) \qquad \Rightarrow \qquad v(2)\cdot u'(2) + u(2)\cdot v'(2) = 1(-3) + 4(5) = \boxed{17}$$

b. $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad \Rightarrow \qquad \frac{v(2)u'(2) - u(2)v'(2)}{\left[v(2)\right]^2} = \frac{1(-3) - 4(5)}{1^2} = \boxed{-23}$$

c. $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$

$$\frac{d}{dx}\left(\frac{v}{u}\right) = \frac{u\frac{dv}{dx} - v\frac{du}{dx}}{u^2} \qquad \Rightarrow \qquad \frac{u(2)v'(2) - v(2)u'(2)}{\left[u(2)\right]^2} = \frac{4(5) - 1(-3)}{4^2} = \boxed{\frac{23}{16}}$$

d. $\dfrac{d}{dx}(3u - 2v + 2uv)$

$$\frac{d}{dx}(3u - 2v + 2uv) = 3\frac{du}{dx} - 2\frac{dv}{dx} + 2\left(v\frac{du}{dx} + u\frac{dv}{dx}\right)$$

$$\Rightarrow \quad \begin{aligned} &3\cdot u'(2) - 2\cdot v'(2) + 2\left[v(2)\cdot u'(2) + u(2)\cdot v'(2)\right] \\ &= 3(-3) - 2(5) + 2\left[1(-3) + 4(5)\right] = -9 - 10 + 34 = \boxed{15} \end{aligned}$$

<u>Tangent Lines</u>

Ex9: Find the value(s) of *x* where the curve has horizontal tangent(s). $f(x) = 5x^3 - 10x^2 + 8$

A horizontal tangent line would have a slope of zero. The slope of a tangent is the derivative of the function. So, to find the horizontal tangent(s), we must set the derivative of the function equal to zero.

$$f'(x) = 10x^2 - 20x \qquad 10x^2 - 20x = 0 \Rightarrow 10x(x - 20) = 0 \Rightarrow \boxed{x = 0, 20}$$

<u>Higher Order Derivatives</u>

n^{th} Derivative: the derivative taken n times; notation: $\frac{d^n y}{dx^n} = f^{(n)}(x)$

Ex10 Find the first 4 derivatives of $f(x) = x^3 - 6x^2 + x - 9$.

1st Derivative: $f'(x) = 3x^2 - 12x + 1$

2nd Derivative: $f''(x) = 6x - 12$

3rd Derivative: $f'''(x) = 6$

4th Derivative: $f^{(4)}(x) = 0$

<u>You Try:</u> Write the equation of the normal line of the curve $f(x) = x^3 - 3x + 1$ at $x = 2$.

<u>QOD:</u> Explain why you do not have to use the product rule with a constant factor.

Sample AP Calculus AB Exam Question(s):

1. If $y = \dfrac{2x+3}{3x+2}$, then $\dfrac{dy}{dx} =$

(A) $\dfrac{12x+13}{(3x+2)^2}$

(B) $\dfrac{12x-13}{(3x+2)^2}$

(C) $\dfrac{5}{(3x+2)^2}$

(D) $\dfrac{-5}{(3x+2)^2}$

(E) $\dfrac{2}{3}$

2. If the line tangent to the graph of the function *f* at the point $(1,7)$ passes through the point $(-2,-2)$, then $f'(-1)$ is

(A) -5
(B) 1
(C) 3
(D) 7
(E) undefined

3. A curve has slope $2x+3$ at each point (x,y) on the curve. Which of the following is an equation for this curve if it passes through the point $(1,2)$?

(A) $y = 5x-3$

(B) $y = x^2+1$

(C) $y = x^2+3x$

(D) $y = x^2+3x-2$

(E) $y = 2x^2+3x-3$

Syllabus Objectives: 2.14 – The student will solve problems involving rates of change. 2.13 – The student will solve problems involving particle motion along a line.

<u>Average Rate of Change</u> $= \dfrac{\text{amount of change}}{\text{time it takes (length of interval)}} = \dfrac{f(x+h)-f(x)}{h}$

<u>Instantaneous Rate of Change</u> at $a = \lim\limits_{x\to 0}\dfrac{f(a+h)-f(a)}{h}$

Note: The above limit should look familiar; it is the **derivative** of *f* at *a*.

Ex1: Find the rate of change of the area *A* of a circle with respect to its radius *r*. Evaluate at $r = 5$ in. What are the appropriate units of this rate?

The formula for the area of a circle is $A = \pi r^2$. The instantaneous rate of change is the derivative of the area with respect to *r*.

$$\frac{d}{dr}(A) = 2\pi r \quad \Rightarrow \quad \left.\frac{dA}{dr}\right|_{r=5} = 2\pi(5) = 10\pi \qquad \text{Units: } 10\pi\ \text{in}^2/\text{in} = \boxed{10\pi\ \text{in}}$$

<u>Motion Along a Line</u>

Position: where an object is after *t* seconds; Notation: $s(t) = f(t)$

Displacement: change in position over time from *t* to Δt; $f(t+\Delta t) - f(t)$ = position after Δt seconds minus the initial position

Average Velocity: change in position divided by change in time; $\dfrac{\text{displacement}}{\text{travel time}} = \dfrac{\Delta s}{\Delta t} = \dfrac{f(t+\Delta t)-f(t)}{\Delta t}$

Instantaneous Velocity: $\lim\limits_{\Delta t\to 0}\dfrac{f(t+\Delta t)-f(t)}{\Delta t}$; Notation: $v(t) = s'(t)$

Note: Velocity can be positive, negative or zero. A positive velocity represents movement to the right (or up); a negative velocity represents movement to the left (or down); and a zero velocity represents standing still.

Speed: a nonnegative value representing how fast an object is going; Notation: speed $= |v(t)|$

Acceleration: the rate of change of the velocity; Notation: $a(t) = v'(t) = s''(t)$

Note: Acceleration can be positive, negative or zero. A positive acceleration represents the object speeding up; a negative acceleration represents the object slowing down; a zero acceleration represents a constant velocity.

Acceleration Due to Gravity (Earth): $g = 32\ \text{ft}/\text{sec}^2$ or $9.8\ \text{m}/\text{sec}^2$

<u>Vertical Motion</u>

Ex2: A rock is thrown vertically upward from the surface of the moon at a velocity of 8 m/s. Its position is represented by the function $s = 8t - 0.8t^2$ m in *t* seconds. (Note: Always label answers with correct units, and round to 3 decimal places.)

a) Find $v(t)$ & $a(t)$.

$v(t) = s'(t) = 8 - 1.6t \qquad a(t) = v'(t) = -1.6$

b) How long does it take for the rock to reach its highest point?

At its highest point, the velocity of the rock is zero. (It must stop to change direction.)

$$v(t) = 8 - 1.6t = 0 \Rightarrow \boxed{t = 5\text{sec}}$$

c) How high did the rock go?

We found in part b) that the rock reached its highest point at 5 seconds. So we will find its position at 5 sec.

$$s(5) = 8(5) - 0.8(5)^2 = \boxed{20 \text{ m}}$$

d) When did the rock reach half its maximum height?

Half of its maximum height is $20/2 = 10$. We will solve for *t* using the position function with $s = 10$.

$$10 = 8t - 0.8t^2 \Rightarrow 0.8t^2 - 8t + 10 = 0 \Rightarrow 8t^2 - 80t + 100 = 0 \Rightarrow 2t^2 - 20t + 25 = 0$$

Solve using the quadratic formula: $t = \dfrac{20 \pm \sqrt{400 - 200}}{4} = \dfrac{20 \pm 10\sqrt{2}}{4} = \dfrac{10 \pm 5\sqrt{2}}{2}$

$$t = \frac{10 - 5\sqrt{2}}{2} \approx \boxed{1.464 \text{ sec}} \quad \text{or} \quad t = \frac{10 + 5\sqrt{2}}{2} \approx \boxed{8.536 \text{ sec}}$$

e) How long was the rock aloft?

The rock will return to the ground when $s = 0$.

$$0 = 8t - 0.8t^2 \Rightarrow 0.8t(10 - t) = 0 \Rightarrow t = 0, \boxed{t = 10 \text{ sec}}$$

Note: $t = 0$ represents the rock on the ground before it was thrown.

Particle Motion

Ex3: A particle moves along a horizontal line so that its position at any time $t \geq 0$ is modeled by the function $s(t) = t^3 - 6t^2 + 9$, where *t* is measured in seconds and *s* is measured in meters.

a) Find the displacement of the particle during the first 2 seconds.

Displacement = change in position $\quad s(2) - s(0) = (2^3 - 6 \cdot 2^2 + 9 \cdot 2) - (9) = \boxed{-7 \text{ m}}$

b) Find the average velocity of the particle during the first 2 seconds.

Average velocity = displacement divided by change in time $\quad \text{Avg Vel} = \dfrac{-7}{2} = \boxed{-\dfrac{7}{2} \text{ m/sec}}$

c) Find the instantaneous velocity of the particle when $t = 2$.

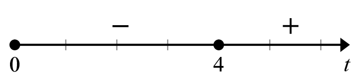
$v(t) = s'(t) = 3t^2 - 12t \qquad v(2) = 3 \cdot 2^2 - 12 \cdot 2 = \boxed{-12 \text{ m/sec}}$

d) Describe the motion of the particle. At what values of *t* does the particle change directions?

The particle is moving left when the velocity is negative, right when the velocity is positive. It changes directions when the velocity is zero.

$v(t) = 3t^2 - 12t = 0 \qquad 3t(t-4) = 0 \qquad t = 0, t = 4$

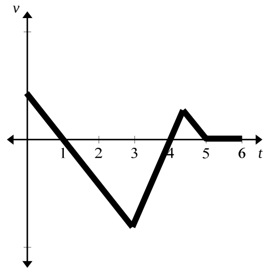
When $3t(t-4) > 0$, the particle is moving right. When $3t(t-4) < 0$, the particle is moving left. We will use a sign chart.



So the particle is moving left for 4 seconds and right after 4 seconds.

The particle changes directions at 4 seconds.

You Try: The graph below shows the velocity in m/sec of an object moving along a line.



a) When does the object reverse directions?

b) When is the object moving at a constant speed?

c) Graph the object's speed for $0 \le t \le 6$.

d) Graph the acceleration of the object, where defined.

QOD: What does it mean for an object to have a negative displacement?

Sample AP Calculus AB Exam Question(s) (taken from the released 2003 MC AP Exam):

1. A particle moves along the *x*-axis so that at time $t \geq 0$ its position is given by $x(t) = 2t^3 - 21t^2 + 72t - 53$. At what time *t* is the particle at rest?

(A) $t = 1$ only

(B) $t = 3$ only

(C) $t = \frac{7}{2}$ only

(D) $t = 3$ and $t = \frac{7}{2}$

(E) $t = 3$ and $t = 4$

2. A particle moves along the *x*-axis so that at any time $t \geq 0$, its velocity is given by $v(t) = 3 + 4.1\cos(0.9t)$. What is the acceleration of the particle at time $t = 4$?

(A) -2.016

(B) -0.677

(C) 1.633

(D) 1.814

(E) 2.978