AP Calculus Notes: Unit 1 – Limits & Continuity

Syllabus Objective: 1.1 – The student will calculate limits using the basic limit theorems.

<u>LIMITS</u> – how the outputs of a function behave as the inputs **approach** some value

<u>Notation</u>: "The limit as x approaches c of $f(x)$" $\qquad \lim_{x \to c} f(x)$

<u>Finding a Limit</u>

I. Table

Ex1: $\lim_{x \to 0} \frac{\sin x}{x}$

Use the table to choose values of x close to zero (from the left and right).

As x approaches 0, it appears the function is approaching 1.

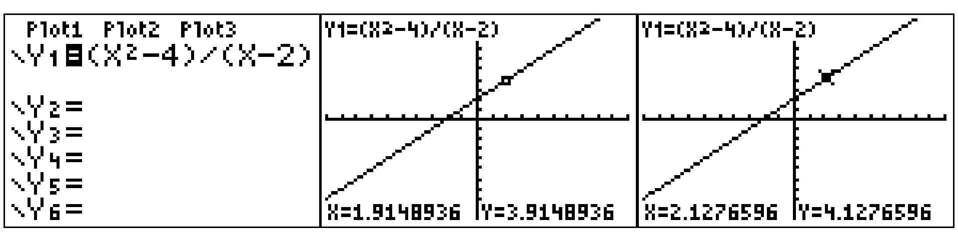
Note: Existence at the point is not relevant when calculating a limit.

$$\lim_{x \to 0} \frac{\sin x}{x} = \boxed{1}$$ **Note: "Special Limit" – must memorize!**

II. Graphically

Ex2: $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

Graph the function and use the TRACE feature to see function values as we approach 2 from the left and right.



As x approaches 2, it appears the function is approaching 4.

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \boxed{4}$$

III. Analytically

a. Direct Substitution

Ex3: $\lim_{x\to 2}\frac{x^2-2x+8}{x+2}$

Substitute $x=2$ into the function: $\lim_{x\to 2}\frac{x^2-2x+8}{x+2}=\frac{(2)^2-2(2)+8}{2+2}=\boxed{2}$

b. Algebraic Manipulation

Ex4: $\lim_{x\to -3}\frac{9-x^2}{x+3}$

We cannot use direct substitution, because of division by zero. So we must simplify the function algebraically.

$$\lim_{x\to -3}\frac{9-x^2}{x+3}=\lim_{x\to -3}\frac{(3-x)\cancel{(3+x)}}{\cancel{(x+3)}}=\lim_{x\to -3}(3-x)=3-(-3)=\boxed{6}$$

Ex5: $\lim_{x\to 0}\frac{\sqrt{1-x}}{x}$

We cannot use direct substitution, because of division by zero. Simplify the function algebraically by rationalizing the numerator.

$$\lim_{x\to 0}\frac{\sqrt{1-x}-1}{x}\cdot\frac{\sqrt{1-x}+1}{\sqrt{1-x}+1}=\lim_{x\to 0}\frac{1-x-1}{x\left(\sqrt{1-x}+1\right)}=\lim_{x\to 0}\frac{-\cancel{x}}{\cancel{x}\left(\sqrt{1-x}+1\right)}$$

$$=\lim_{x\to 0}\frac{-1}{\sqrt{1-x}+1}=\frac{-1}{\sqrt{1-0}+1}=\boxed{-\frac{1}{2}}$$

<u>Nonexistent Limit</u>

Ex6: $\lim_{x\to -3}\frac{x^3-1}{x+3}$

We cannot use direct substitution, because of division by zero. Factoring the numerator, we have:

$\lim_{x\to -3}\frac{x^3-1}{x+3}=\lim_{x\to -3}\frac{(x-1)\left(x^2+x+1\right)}{x+3}$ This does not simplify to avoid division by zero.

So, $\lim_{x\to -3}\frac{x^3-1}{x+3}$ <u>does not exist</u>.

One-sided Limits

1. Right-Hand Limit: the limit of *f* as *x* approaches *c* from the right. $\lim_{x \to c^+} f(x)$

Ex7: Find $\lim_{x \to 2^+} f(x)$, when $f(x) = \begin{cases} x^2 & x \le 2 \\ 3x - 1 & x > 2 \end{cases}$

Because we are approaching from the right, we must substitute into the piece of the piecewise functions that represents values of *x* greater than 2.

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3x - 1 = 3(2) - 1 = \boxed{5}$ Note: The limit value differs from the function value at 2.

2. Left-Hand Limit: the limit of *f* as *x* approaches *c* from the left. $\lim_{x \to c^-} f(x)$

Ex8: $\lim_{x \to 2^-} \sqrt{x-2}$

We cannot use direct substitution, because we cannot approach 2 from the left. The domain of the function $f(x) = \sqrt{x-2}$ is $[2, \infty)$. So $\lim_{x \to 2^-} \sqrt{x-2}$ does not exist.

Two-sided Limit

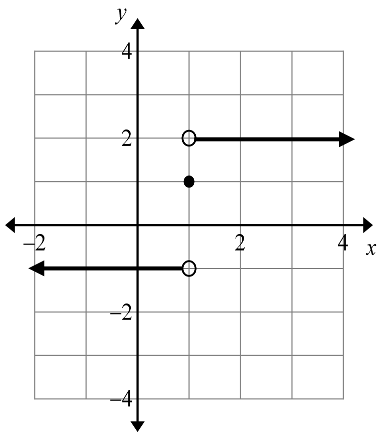
$$\lim_{x \to c} f(x)$$

** $f(x)$ has a limit as *x* approaches *c* if and only if the right and left hand limits at *c* exist and are equal**

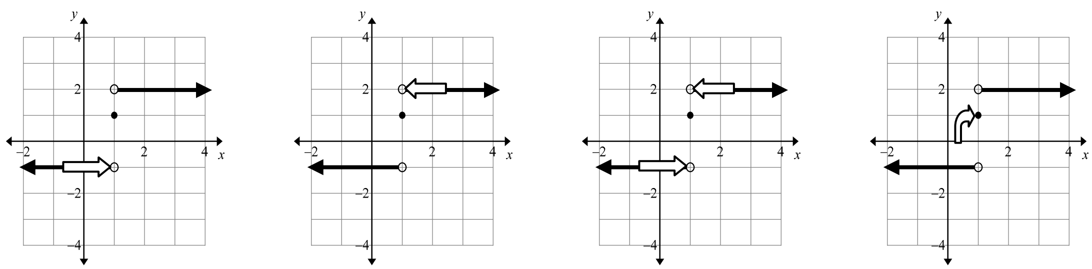
Limits That Fail to Exist

- Right and Left Behavior Differs: $\lim_{x \to c^+} f(x) \ne \lim_{x \to c^-} f(x)$
- Unbounded: $\lim_{x \to c} f(x) = -\infty$ or $\lim_{x \to c} f(x) = \infty$
- Oscillating: For example, $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$. Show by graphing.

Ex9: Use the graph below of $f(x)$ to answer the following questions.



a. $\lim_{x\to 1^-} f(x)$ b. $\lim_{x\to 1^+} f(x)$ c. $\lim_{x\to 1} f(x)$ d. $f(1)$



$\lim_{x\to 1^-} f(x) = \boxed{-1}$ $\lim_{x\to 1^+} f(x) = \boxed{2}$ $\lim_{x\to 1} f(x) = \boxed{\text{does not exist}}$ $f(1) = \boxed{1}$

PROPERTIES OF LIMITS:

If $L, M, c,$ and k are real numbers and

$\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

1. Sum and Difference Rules: $\lim_{x\to c}(f(x) \pm g(x)) = L \pm M$

   The limit of the sum or difference of two functions is the sum or difference of their limits.

2. Product Rule: $\lim_{x\to c}(f(x)\cdot g(x)) = L\cdot M$

   The limit of a product of two functions is the product of their limits.

3. Constant Multiple Rule: $\lim_{x\to c}(k\cdot f(x)) = k\cdot L$

   The limit of a constant times a function is the constant times the limit of the function.

4. Quotient Rule: $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$

   The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

5. Power Rule: If r and s are integers, $s \neq 0$, then

   $\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$ provided that $L^{r/s}$ is a real number.

Ex10: Find $\lim_{x\to 0}\frac{\tan x}{x}$.

Using direct substitution, we divide by zero. So try algebraic manipulation. We can rewrite $\tan x$ as $\frac{\sin x}{\cos x}$.

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{x} = \lim_{x\to 0}\frac{\sin x}{x\cos x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}$$

Using the product rule, $\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x} = 1\cdot\frac{1}{1} = \boxed{1}$

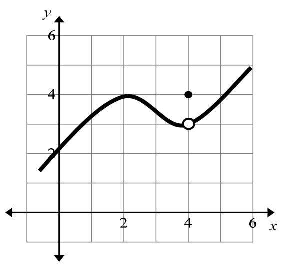
<u>You Try:</u> Calculate the limit. $\lim_{x\to 2}\frac{t^2-3t+2}{t^2-4}$

<u>QOD:</u> Explain how left and right-hand limits relate to two-sided limits.

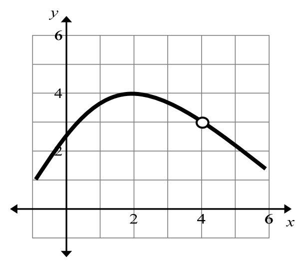
Sample AP Calculus AB Exam Question(s) (taken from the released 2003 MC AP Exam):

For which of the following does $\lim_{x \to 4} f(x)$ exist? (Note: Each function shown is $f(x)$.)

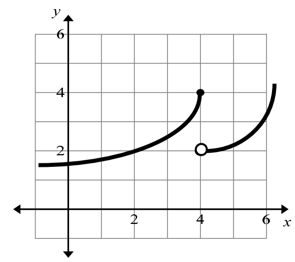
I.



II.



III.



(A) I only
(B) II only
(C) III only
(D) I and II only
(E) I and III only

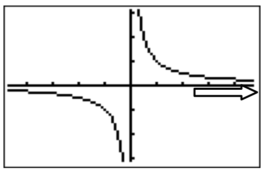
Syllabus Objective: 1.1 – The student will calculate limits using the basic limit theorems. (Limits involving infinity.)

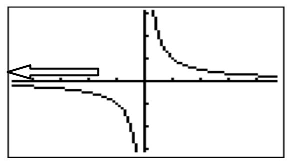
Notation:

∞ increasingly far to the right (or up)

$-\infty$ increasingly far to the left (or down)

Ex1: Graph: $f(x)=\frac{1}{x}$. Find $\lim_{x\to\infty}\frac{1}{x}$ and $\lim_{x\to-\infty}\frac{1}{x}$.





$\lim_{x\to\infty}\frac{1}{x}=\boxed{0}$ $\lim_{x\to-\infty}\frac{1}{x}=\boxed{0}$

Definition:

The line *y=b* is a **horizontal asymptote** of the graph of a function *y = f(x)* if either

$$\lim_{x\to\infty} f(x)=b \text{ or } \lim_{x\to-\infty} f(x)=b$$

Ex2: Find any horizontal asymptote(s) of the function $f(x)=\frac{3x^2-x+5}{x^2-4}$.

Find $\lim_{x\to\infty}\frac{3x^2-x+5}{x^2-4}$ (or $\lim_{x\to-\infty}\frac{3x^2-x+5}{x^2-4}$). To calculate this limit, begin by dividing each term by the highest power of *x*, which in this case is x^2.

$$\lim_{x\to\infty}\frac{3x^2-x+5}{x^2-4}=\lim_{x\to\infty}\frac{\frac{3x^2}{x^2}-\frac{x}{x^2}+\frac{5}{x^2}}{\frac{x^2}{x^2}-\frac{4}{x^2}}=\lim_{x\to\infty}\frac{3-\frac{1}{x}+\frac{5}{x^2}}{1-\frac{4}{x^2}}=\frac{3-0+0}{1-0}=3$$

So the horizontal asymptote is $\boxed{y=3}$.

Note: When finding $\lim_{x\to\infty}\frac{f(x)}{g(x)}$, you can use the following guidelines.

- The $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\infty$ if the degree of $f(x)$ is greater than the degree of $g(x)$.

- The $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=0$ if the degree of $f(x)$ is less than the degree of $g(x)$.

- If the degree of $f(x)$ is equal to the degree of $g(x)$, then $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}$ is equal to the ratio of the lead coefficients.

<u>Infinite Limits</u>

When *f* approaches infinity as *x* approaches *a*, the limit does not exist, and is called **unbounded**.

<u>Definition</u>:

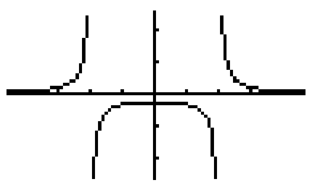
The line *x* = *a* is a **vertical asymptote** of the graph of a function *y* = *f(x)* if either

$\lim\limits_{x\to a^+} f(x)=\pm\infty$ or $\lim\limits_{x\to a^-} f(x)=\pm\infty$

Ex3: Find the vertical asymptote(s) of $f(x)=\dfrac{1}{x^2-4}$.

A fraction will approach $\pm\infty$ if the denominator of the fraction approaches zero. To find the vertical asymptote(s), set the denominator equal to zero and solve for *x*.

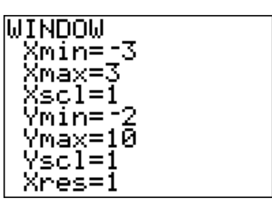
$x^2-4=0 \Rightarrow \boxed{x=-2, x=2}$

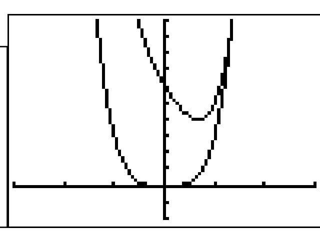
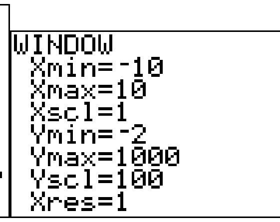
Take a look at the graph: 

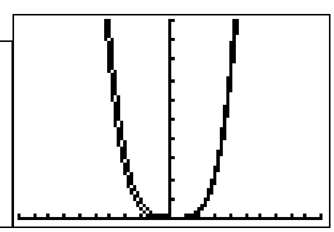
$\lim\limits_{x\to -2^-}\dfrac{1}{x^2-4}=\infty$ $\lim\limits_{x\to -2^+}\dfrac{1}{x^2-4}=-\infty$ $\lim\limits_{x\to 2^-}\dfrac{1}{x^2-4}=-\infty$ $\lim\limits_{x\to -2^+}\dfrac{1}{x^2-4}=\infty$

<u>End Behavior Models:</u> As *x* becomes very large, a more complicated function can be modeled by a simpler one.

To see this, graph $f(x)=3x^4-2x^3+3x^2-5x+6$ and $g(x)=3x^4$ Note: $\lim\limits_{x\to\pm\infty}\dfrac{f(x)}{g(x)}=1$





For larger values of *x*, the graphs appear alike.

Finding End Behavior Models:

Right End Behavior Model
Find g such that $\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$

Left End Behavior Model
Find g such that $\lim_{x\to-\infty}\frac{f(x)}{g(x)}=1$

Ex4: Find an end behavior model for $f\left(x\right)=\frac{3x^5-2x^2+5x+1}{2x^3-5}$.

End behavior model for the numerator: $3x^5$ because $\lim_{x\to\pm\infty}\frac{3x^5-2x^2+5x+1}{3x^5}=1$

End behavior model for the denominator: $2x^3$ because $\lim_{x\to\pm\infty}\frac{2x^3}{2x^3-5}=1$

End behavior model for *f*: $\frac{3x^5}{2x^3}=\boxed{\frac{3}{2}x^2}$

You Try: Find $\lim_{x\to\infty}f\left(x\right)$ and $\lim_{x\to-\infty}f\left(x\right)$ when $f\left(x\right)=\frac{|x|}{x}$.

QOD: Is an infinite limit nonexistent? Explain your answer.

Sample AP Calculus AB Exam Question(s) (taken from the released 2003 MC AP Exam):

1. For $x \geq 0$, the horizontal line $y = 2$ is an asymptote for the graph of the function *f*. Which of the following statements must be true?

   (A) $f(x) \neq 2$ for all $x \geq 0$

   (B) $f(0) = 2$

   (C) $f(2)$ is undefined

   (D) $\lim\limits_{x \to 2} f(x) = \infty$

   (E) $\lim\limits_{x \to \infty} f(x) = 2$

2. $\lim\limits_{x \to \infty} \dfrac{x^3 - 2x^2 + 3x - 4}{4x^3 - 3x^2 + 2x - 1} =$

   (A) 4

   (B) 1

   (C) $\dfrac{1}{4}$

   (D) 0

   (E) -1

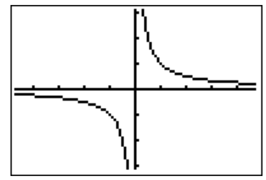
Syllabus Objective 2.1 – The student will analyze the continuity of an elementary function.

Continuous Function (on an interval): a function whose outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between on the given interval

(i.e. – The graph can be traced without lifting your pencil!)

Continuous Function: a function that is continuous at every point of its domain

Ex1: Graph the function $f(x)=\dfrac{1}{x}$. Determine an interval in which the function is not continuous. Is $f(x)=\dfrac{1}{x}$ a continuous function?



In the interval $[-2,2]$, the outputs do not take on all values in between $f(-2)$ and $f(2)$ for any interval that includes $x = 0$. So the function is **not** continuous in the interval $[-2,2]$. (Or any interval containing 0.)

$f(x)$ **is** a continuous function, because it is continuous on every point in its domain. ($x = 0$ is not in its domain.)

Definition: Continuity at a Point

1. $f(x)$ is continuous at an INTERIOR POINT c of its domain if $\lim_{x\to c} f(x) = f(c)$

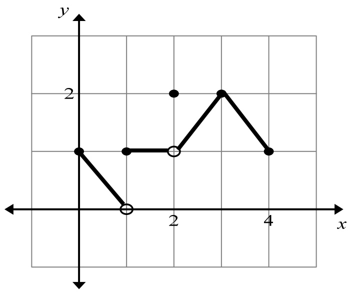
   a. $f(c)$ is defined (exists)

   b. $\lim_{x\to c} f(x)$ exists

   c. $\lim_{x\to c} f(x) = f(c)$

2. $f(x)$ is continuous at a LEFT ENDPOINT a or a RIGHT ENDPOINT b of its domain if $\lim_{x\to a^+} f(x) = f(a)$ or $\lim_{x\to b^-} f(x) = f(b)$ respectively.

Ex2: Find the points of discontinuity of the function shown in the graph.



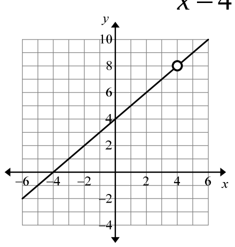
$\boxed{x=1}$**:** $\lim_{x\to1} f(x)$ does not exist because the left- and right-hand limits are not equal.

$\boxed{x=2}$: $\lim_{x\to2} f(x)=1\neq f(2)=2$

Types of Discontinuity:

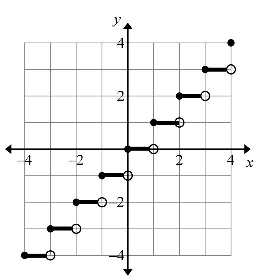
1. Removable (hole)

   Ex3: $f(x)=\dfrac{x^2-16}{x-4}$ at $x=4$



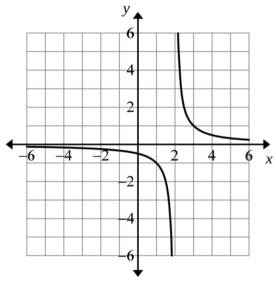
2. Jump – one-sided limits exist, but are different

   Ex4: $f(x)=[\![x]\!]$ (Greatest Integer Function), at every integer value of x



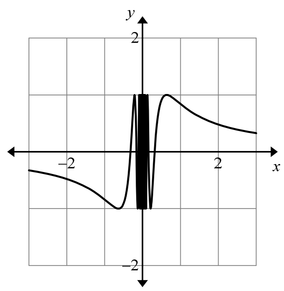
3. Infinite (occurs when there is a vertical asymptote)

Ex5: $f(x)=\dfrac{1}{x-2}$ at $x=2$



4. Oscillating

Ex6: $f(x)=\sin\left(\dfrac{1}{x}\right)$ at $x=0$



Teacher Note: Have students graph on calculator and zoom in on $x = 0$ to watch the oscillation.

<u>Removing a Discontinuity</u>: a "hole" in a graph is a removable discontinuity

Ex7: Identify the point of discontinuity in the function $f(x)=\dfrac{x^2-3x-4}{x+1}$. Then write an *extended* function that is continuous at this point ("remove" the discontinuity).

Point of Discontinuity: $\begin{aligned} x+1&=0 \\ x&=-1 \end{aligned}$

$f(x)=\dfrac{x^2-3x-4}{x+1}$ is the graph of $f(x)=\dfrac{(x-4)\cancel{(x+1)}}{\cancel{(x+1)}} \Rightarrow f(x)=x-4$ with a "hole" at $x=-1$.

To fill the hole, we must find $f(-1)=-1-4=-5$.

Extended function: $f(x)=\begin{cases} \dfrac{x^2-3x-4}{x+1}, & x\neq -1 \\ -5, & x=-1 \end{cases}$

Properties of Continuous Functions: If f and g are continuous at $x = c$, then the following are continuous at $x = c$.

- Sum: $f + g$
- Difference: $f - g$
- Product: $f \cdot g$
- Constant Multiple: $k \cdot f$, for any real number k
- Quotient: $\frac{f}{g}$, provided $g(c) \neq 0$

Composite Functions: If f is continuous at c and g is continuous at $f(c)$, then the composite $f \circ g$ is continuous at c.

Intermediate Value Theorem: A function $y = f(x)$ that is continuous on a closed interval $[a,b]$ takes on every value between $f(a)$ and $f(b)$.

> **Ex8**: The local weatherman reported that the low temperature today was 67° and the high was 89°. From this information, can you guarantee that sometime during the day the temperature was 73°? Explain.

YES. Temperature is continuous. It cannot jump from one degree to another without hitting every value in between. So by the *Intermediate Value Theorem*, the temperature had to be 73° at some point during the day.

You Try: Graph a function $f(x)$ for which $\lim_{x \to 1} f(x)$ exists, but is not continuous at $x = 1$.

QOD: Can a continuous function have a point of discontinuity? Explain.

Sample AP Calculus AB Exam Question(s):

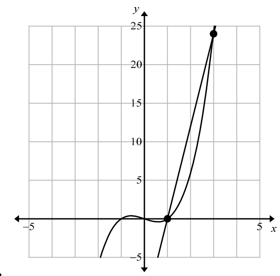
On which of the following intervals is $f(x) = \frac{1}{x^2}$ not continuous?

(A) $(0,\infty)$
(B) $[0,\infty)$
(C) $(0,2)$
(D) $(1,2)$
(E) $(-2,-1)$

Syllabus Objective: The student will calculate limits using the basic limit theorems. (Applications to instantaneous rate of change and slope)

Average Rate of Change $= \dfrac{\text{amount of change}}{\text{time it takes (length of interval)}}$ Slope $= m = \dfrac{y_1 - y_2}{x_1 - x_2}$

Ex1: Find the average rate of change of $f(x) = x^3 - x$ over the interval [1, 3].



Finding the average rate of change is the same as finding the *slope* of the secant line.

Average rate of change = $\dfrac{f(3) - f(1)}{3-1} = \dfrac{\left[(3)^3 - 3\right] - \left[(1)^3 - 1\right]}{2} = \dfrac{24}{2} = \boxed{12}$

Pierre Fermat (1629)

1. Start with the slope of a secant through *P* and a point *Q* nearby.
2. Find the limiting value of the secant slope as *Q* approaches *P*.
3. This is the slope of the curve at *P* and the slope of the tangent line to the curve at *P*.

TANGENT TO A CURVE – the line through *P* with the slope as calculated above

The SLOPE OF THE CURVE $y = f(x)$ at the point $P(a, f(a))$ is $m = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

Note: As *h* approaches 0, the two points approach one point. The slope of the curve at point *P* is the same as the slope of the tangent line at point *P*.

Ex2: Find the slope of the parabola $y = x^2$ at the point *P*(3, 9). Write an equation for the tangent line at *P*.

Slope: $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h} = \lim\limits_{h \to 0} \dfrac{(3+h)^2 - (3)^2}{h} = \lim\limits_{h \to 0} \dfrac{9 + 6h + h^2 - 9}{h} = \lim\limits_{h \to 0} \dfrac{6h + h^2}{h} = \lim\limits_{h \to 0} 6 + h = \boxed{6}$

Equation of tangent line: $\boxed{y - 9 = 6(x - 3)}$

NORMAL LINE: the line perpendicular to a tangent line at a given point

Ex3: Write the equation of the normal line in the example above.

Perpendicular to the tangent at *P*: $\boxed{y - 9 = -\dfrac{1}{6}(x - 3)}$

INSTANTANEOUS SPEED: an object's speed at any given time

Ex4: An object is dropped from the top of a 50-ft building. Its height above the ground after *t* seconds is $50-4.9t^2$. How fast is it falling 1 second after it is dropped?

We must find the instantaneous speed, or rate of change, at 1 second.

$$\lim_{h\to 0}\frac{50-4.9(1+h)^2-\left(50-4.9(1)^2\right)}{h}=\lim_{h\to 0}\frac{50-4.9-9.8h-4.9h^2-50+4.9}{h}=\lim_{h\to 0}\frac{-9.8h-4.9h^2}{h}=\lim_{h\to 0}-9.8-4.9h$$

$=-9.8$ Note: Speed must be nonnegative. So the speed of the object at $t = 1$ is $\boxed{9.8\text{ ft/sec}}$.

You Try:

1. Find the average rate of change of the function $f(x)=2+\cos x$ on the interval $[0,\pi]$.
2. Write the equations of the tangent and normal lines for $y=x^2-4x$ at $x=1$.

QOD: How is the slope of a tangent line derived from the slope of a secant line?

Sample AP Calculus AB Exam Question(s) (taken from the released 2003 MC AP Exam):

Let *f* be the function defined by $f(x)=4x^3-5x+3$. Which of the following is an equation of the line tangent to the graph of *f* at the point where $x=-1$?

(A) $y=7x-3$

(B) $y=7x+7$

(C) $y=7x+11$

(D) $y=-5x-1$

(E) $y=-5x-5$