Geometry Unit 2 – Notes
Logic, Reasoning and Proof

***Review Vocab.**: Complementary, Supplementary and Vertical angles.*

Syllabus Objective: 2.2 - The student will justify conjectures and solve problem using inductive reasoning.

Conjecture – an unproven statement that is based on observations. {Could end up true or false}

Inductive Reasoning – Reasoning that uses specific examples to arrive at a logical conclusion. Conclusions are not logical certainty, but the best 'guess' based on what has been observed. Given a pattern of numbers and assuming that it continues is an application of inductive reasoning.

Examples of Conjectures: **State the conclusions that can be made from the observations:**

a) The postman delivers mail Monday through Thursday at 1:00 PM.
We could conclude that he will deliver mail at 1:00 PM on Friday.
?: What happens if he has a flat tire? Or an accident?

b) It has been hot and sunny for 5 days straight.
We might conclude that tomorrow will be hot and sunny.
?: Could there be a freak cold snap?

Examples of Inductive Reasoning:

a) Find the next item in the sequence:
Movie show times: 8:30 AM, 9:45 AM, 11:00 AM, 12:15 PM, … 1:30 PM
?: How far apart are the movies show times spaced? 1 hour and 15 minutes.

b) Predict the next two numbers in the patterns:
2, 4, 7, 11, … ______, ______
$3, 1, \frac{1}{3}, \frac{1}{9}, \ldots$ ______, ______

Semester Exam Review (Practice Test 08-09)

In the scientific method, after one makes a conjecture, one tests the conjecture. What type of reasoning is used?

A. conclusive

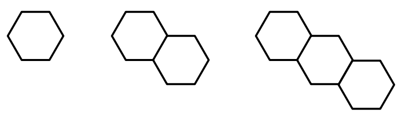
B. deductive

C. inductive

D. scientific

Semester Exam Review (Practice Test 08-09)

In the pattern below, the sides of each regular hexagon have a length of 1 unit.



What is the perimeter of the 5^{th} figure?

A. 18 units

B. 22 units

C. 26 units

D. 30 units

Semester Exam Review (Practice Test 08-09):

What is the n^{th} term of the sequence
1, 4, 9, 16, 25 …?

A. $2n-1$

B. $n+3$

C. n^2

D. $3n^2$

Semester Exam Review (Practice Test 08-09):

Geometric figures are displayed on a computer screen in the following order: triangle, concave quadrilateral, convex pentagon, concave hexagon. Using inductive reasoning, what prediction can be made about the next figure?

A. It will be a concave heptagon.

B. It will be a convex heptagon.

C. It will be a convex polygon, but the type cannot be predicted.

D. It will be a polygon, but no other details about it can be predicted.

Sample Nevada High School Proficiency Exam question(s): Taken from the 2009 Instructional Materials for the NHSPE provided by the Nevada Department of Education

1. The first four terms of a sequence are shown below.

   $\frac{1}{2} \quad \frac{1}{8} \quad \frac{1}{18} \quad \frac{1}{32}$

   The sequence continues. What is the seventh term of the sequence?

   A $\frac{1}{256}$

   B $\frac{1}{98}$

   C $\frac{1}{64}$

   D $\frac{1}{60}$

2. The first five terms of a sequence are shown below.

   4 10 28 82 244

   The sequence continues. What is the sixth term of the sequence?

   A 368
   B 486
   C 730
   D 732

Syllabus Objective: 2.4 – The student will distinguish between the hypothesis and conclusion of an implication.
Syllabus Objective: 2.5 – The student will write an implication as an if-then statement.

Conditional Statement (Implication) – A type of logical statement that has two parts, a hypothesis and a conclusion. Conditional statements are often written in 'if-then' format.

Hypothesis: the independent condition.
Conclusion: the condition dependent upon the hypothesis.

Example: Identify the hypothesis and conclusion of the conditional:
If an animal is a duck, then it is a bird.
Hypothesis: an animal is a duck.
Conclusion: it is a bird.

Example: Write the statement in "if-then" form:
"*All numbers that are divisible by 4 are also divisible by 2.*"
If a number is divisible by 4 then it is also divisible by 2.

Semester Exam Review (Practice Test 08-09):

All donks are widgets. Which statement can be written using the rules of logic?

A. A donk is a widget if and only if it is an object.
B. An object is a donk if and only if it is a widget.
C. If an object is a widget, then it is a donk.
D. If an object is a donk, then it is a widget.

Examples: Write each implication as an if-then statement:

a) The sum of the measures of two supplementary angles is 180°.
- If the sum of the measures of two angles is 180°, then they are supplementary.

b) Two angles are complementary if the sum of their measures is 90°.
- If the sum of the measures of two angles is 90°, then they are complementary.

*Notice the placement of the word 'if' in the second example changes the customary placement of '*if-then*' in the sentence. Look for other examples that affect the sequencing of the statement.

The conclusion is dependent upon the hypothesis. The hypothesis is independent. This is analogous to the x-y relationship in functions.

Syllabus Objective: 2.8 – The student will find counterexamples to disprove mathematical statements.

Counterexample – An example used to show that a given statement is not always true. {All you need is one. One counterexample disproves the statement.}

Examples: Give a counterexample that proves each statement is false:

a) If n is a real number, then $-n$ is negative.

Counterexample: (possible answer) If $n = -5$ (a real number), then $-n = 5$ (a positive number).

b) If n is a prime number, then $n + 1$ is not prime.

Counterexample: (possible answer) If $n = 2$ (a prime number), then $n + 1 = 3$ (also prime).

c) If the area of a rectangle is 20 ft^2, then its length is 10 feet and its width is 2 feet.

Counterexample: (possible answer) If the length is 5 feet and the width is 4 feet then the area would still be 20 ft^2.

Semester Exam Review (Practice Test 08-09):

Which is a valid counterexample of the converse of the statement: *If Hedley lives in North Las Vegas, then he lives in Nevada*?

A. Hedley lives in Phoenix.

B. Hedley lives in California.

C. Hedley lives in Reno.

D. Hedley lives in the United States.

Syllabus Objective: 2.7 – The student will write and analyze converse, inverse, and contrapositive of a statement.

Symbolic notation of conditional statements:

- p: 'hypothesis condition'
- q: 'conclusion condition'
- $\rightarrow$ 'implies'
- $\leftrightarrow$ 'if and only if'
- $\sim$ 'not' or negation

If p, then q …	$p \rightarrow q$	… p implies q
If not q, then not p …	$\sim q \rightarrow \sim p$	
p if and only if q …	$p \leftrightarrow q$	

Converse - a form of a conditional statement created by 'switching' the hypothesis and conclusion. $q \rightarrow p$ {Think converse tennis shoes, they MOVE!}

Inverse - a form of a conditional statement created by 'negating' both the hypothesis and conclusion. $\sim p \rightarrow \sim q$ {Like opposites, in Algebra}

Contrapositive - a form of a conditional statement created by switching AND negating the hypothesis and conclusion. $\sim q \rightarrow \sim p$ {Long word, do the most, move and opposite}

Example: Write the converse, inverse and contrapositive and determine the truth value of each statement.

Conditional Statement: If an animal is a duck, then it is a bird. (True)

Converse: If an animal is a bird, then it is a duck. (False)

Inverse: If an animal is NOT a duck, then it is NOT a bird. (False)

Contrapositive: If an animal is NOT a bird, then it is NOT a duck. (True)

Semester Exam Review (Practice Test 08-09):

Which statement is the inverse of:
If x = 5, then x > 3?

A. If $x = 3$, then $x < 5$.

B. If $x \leq 3$, then $x \neq 5$.

C. If $x > 3$, then $x = 5$.

D. If $x \neq 5$, then $x \leq 3$.

Semester Exam Review (Practice Test 08-09):

Which is the contrapositive to the statement: ***If n is odd, then $n^2 + 2n + 1$ is even.***

A. If $n^2 + 2n + 1$ is odd, then n is even.

B. If $n^2 + 2n + 1$ is even, then n is odd.

C. If n is even, then $n^2 + 2n + 1$ is odd.

D. If n is even, then $n^2 + 2n + 1$ is even.

Logically equivalent statements - two conditional statements which are both true (or both false.) A conditional statement and its contrapositive are logically equivalent. As is the inverse & converse.

Syllabus Objective: 2.3 – The student will differentiate between deductive and inductive reasoning.

Remember:
Inductive Reasoning – Reasoning that uses specific examples to arrive at a logical conclusion. Conclusions are not logical certainty, but the best 'guess' based on what has been observed. Given a pattern of numbers and assuming that it continues is an application of inductive reasoning.

Deductive Reasoning – Reasoning that uses facts, rules, definitions, or properties to reach logical conclusions.

Example: Use deductive reasoning to make a conclusion:
It is stated that two angles are congruent. From the definition of congruent we can assert that their measures are equal.
(This is the type of reasoning used in two-column proofs.)

Law of Detachment: If $p \rightarrow q$ is true and p is true, then q is true.

- Conditional statements do NOT imply that any individual condition exists or is true, only that there is a relationship between the two conditions.
- Detachment applies to a singular conditional statement and the existence (or truth) of the hypothesis condition.

If "$p \rightarrow q$" is true and p exists, then q exists.
If an animal is a duck, then it is a bird.
My pet is a duck.
Therefore, my pet is a bird.

Example: If you tease your little brother/sister, then your parents will be upset with you. You start teasing your sibling, what will the outcome be?

→ Your parents will be upset with you.

?: If your parents are upset with you, does that mean you were teasing your little brother/sister?

→ No, there are probably several reasons why your parents could be upset with you.

Law of Syllogism

If $p \to q$ and $q \to r$ are both true conditionals, then $p \to r$ is also true.

- Syllogism is a 'transitive' process linking two conditional statements in order to form a third conditional statement.
- Like conditional statements, Syllogism does NOT imply that any condition exists or is true.
- Syllogism applies to two conditional statements and allows us to create a third conditional statement by linking the first two.

If $p \to q$ and $q \to r$, then $p \to r$.

If an animal is a duck, then it is a bird.

If an animal is a bird, then it has feathers.

$\therefore$

If an animal is a duck, then it has feathers.

Note:

This symbol "$\therefore$" is a mathematical symbol that represents the word "therefore."

Example: State the conclusion that can be reached from the given statements:

If you do your chores, then you can go out with friends this weekend.

If you go out with your friends, then you will go to the movies.

Therefore…

If you do your chores, then you will go to the movies.

Syllabus Objective: 2.1 - The student will differentiate among definitions, postulates, corollaries, and theorems.

Definition - a statement defining an independent/dependent relationship between two conditions, the *hypothesis* and *conclusion*. They are often written in the 'if-then' form and can be written as a true biconditional. (See definition below.)

Postulate (axiom) – is a statement (basic assumption) assumed to be true without proof.

Theorem – is a statement that has to be proved.

Corollary – is a special case of a theorem. Follows from information proven in the theorem, an extension of what has already been proven or a special case.

Syllabus Objective: 2.6 – The student will analyze *conditional* or *biconditional* statements.

Biconditional Statement - a conditional statement that is logically equivalent to its converse. A biconditional is a true conditional statement whose converse is also true.

- Both statements can be summarized into one statement using the phrase 'if and only if'.
- Geometric definitions are biconditional statements.

Example: Write the definition as a conditional and its converse. Then combine them as a true biconditional.

Perpendicular lines: Two lines are perpendicular if and only if they intersect to form a right angle.

→ If two lines are perpendicular, then they intersect to form a right angle.

→ If two lines intersect to form a right angle, then they are perpendicular lines.

→ Two lines are perpendicular if and only if they intersect to form a right angle.

Syllabus Objective: 2.9 – The student will write algebraic proofs.

Reasoning with Algebraic Properties

- Addition Property of Equality: If $a = b$, then $a + c = b + c$.
- Subtraction Property of Equality: If $a = b$, then $a - c = b - c$.
- Multiplication Property of Equality: If $a = b$, the $ac = bc$.
- Division Property of Equality: If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$.
- Reflexive Property of Equality: If a is a real number, then $a = a$.
- Symmetric Property of Equality: If $a = b$, then $b = a$.
- Transitive Property of Equality: If $a = b$ and $b = c$, then $a = c$.
- Substitution Property of Equality: If $a = b$, then a can be substituted in for b in any expression.
- Distributive Property: $a(b + c) = ab + ac$

> It is often advantageous to show the Addition Property of Equality as follows:
> If $a = b$ and $c = d$, then $a + c = b + d$.
>
> This is more typical of how the property will be used in the context of writing a proof.
>
> It is worth discussing how the Subtraction and Division Properties are special cases of the Addition and Multiplication properties where c is either negative or a reciprocal value, respectively.

Example: Write an algebraic proof:

Given: $\frac{2(5x-1)}{8} = 6$

Prove: $x = 5$

Statements	Reasons
1) $\frac{2(5x-1)}{8} = 6$	**1)** Given
2) $2(5x-1) = 48$	**2)** Multiplication Property of Equality
3) $10x - 2 = 48$	3) Distributive Property
4) $5x = 25$	**4)** Addition Property of Equality
5) $x = 5$	**5)** Division Property of Equality

Semester Exam Review (Practice Test FR 08-09):

Given: $(2x)(x+11)=2(x-3)(x+7)$

Prove: $x=-3$

Supply reasons for each step.

Properties of Equality

	Segment Length	Angle Measure
Reflexive	$AB = AB$	$m\angle A = m\angle A$
Symmetric	If $AB = CD$, then $CD = AB$	If $m\angle A = m\angle B$, then $m\angle \mathbf{B} = m\angle A$
Transitive	If $AB = CD$ and $CD = EF$, then $AB = EF$	If $m\angle A = m\angle B$ and $m\angle B = m\angle C$, then $m\angle A = m\angle C$

Properties of Segment Congruence Theorem: Segment congruence is reflexive, symmetric, and transitive.

- Reflexive: If $\overline{AB}$ is a segment, then $\overline{AB} \cong \overline{AB}$.
- Symmetric: If $\overline{AB} \cong \overline{CD}$ then $\overline{CD} \cong \overline{AB}$.
- Transitive: $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$ then $\overline{AB} \cong \overline{EF}$.

Properties of Angle Congruence Theorem: Angle congruence is reflexive, symmetric, and transitive.

- Reflexive: If $\angle ABC$ is an angle, then $\angle ABC \cong \angle ABC$.
- Symmetric: If $\angle ABC \cong \angle XYZ$ then $\angle XYZ \cong \angle ABC$.
- Transitive: If $\angle ABC \cong \angle XYZ$ and $\angle XYZ \cong \angle PQR$, then $\angle ABC \cong \angle PQR$.

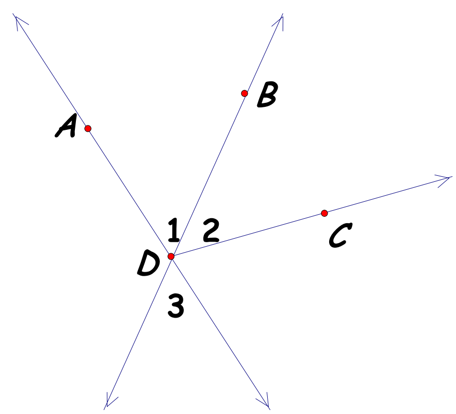
Syllabus Objective: 2.10 – The student will write a formal deductive proof.

When writing geometric proofs students should be required to follow certain minimum guidelines:

1) State the given and what is to be proven.
2) Draw a diagram and label appropriately.
3) For a two column proof make a T chart/table.
4) Label the left column "Statements."
5) Label the right column "Reasons."
6) Complete the chart by writing statements and corresponding reasons that lead to what needs to be proved.

<u>Example:</u> **<u>Write a two-column proof:</u>**

Prove that if $\overrightarrow{DB}$ bisects $\angle ADC$, then $\angle 2 \cong \angle 3$.



Given: $\overrightarrow{DB}$ **bisects** $\angle ADC$

Prove: $\angle 2 \cong \angle 3$

Statements	Reasons
1) $\overrightarrow{DB}$ bisects $\angle ADC$	1) Given
2) $\angle 1 \cong \angle 2$	2) Def. of bisect
3) $\angle 1$ and $\angle 3$ are vertical angles	3) Def. of V. A
4) $\angle 3 \cong \angle 1$	4) V.A. Th
5) $\angle 3 \cong \angle 2$	5) Transitive Prop. of Cong.
6) $\angle 2 \cong \angle 3$	6) Symmetric Prop. of Cong.

Syllabus Objective: 2.11 - The student will write proofs related to segments and angles.

Students should write formal, two-column proofs using the following postulates, properties and/or theorems in conjuncture with definitions.

Postulates:

- Through any two points there is exactly one line.
- A line contains at least two points.
- If two lines intersect, then their intersection is one point.
- Through any three noncollinear points there exists exactly one plane.
- A plane contains at least three noncollinear points.
- If two points lie in a plane, then the line that contains the points lies in the plane.
- If two planes intersect, then their intersection is a line.

Theorems:

Right Angle Congruence Theorem: All right angles are congruent. {Rt. $\angle \cong$ Th.}

Congruent Complements Theorem: If two angles are complementary to the same angle, or to congruent angles, then the angles are congruent. {$\cong$comp. Th.}

Congruent Supplements Theorem: If two angles are supplementary to the same angle, or to congruent angles, then the angles are congruent. {$\cong$supp. Th.}

Linear Pair Postulate: If two angles form a linear pair, then the angles are supplementary. {L.P. Post.}

Vertical Angles Theorem: Vertical angles are congruent. {V.A. Th.}

When writing a 2-Column Proof, it may be helpful to write justifications for each statement in conditional form. The conclusion condition of each justification must 'mirror' the statement being justified.

Ex) The statement, $\angle XYZ \cong \angle PQR$ must be justified by a conditional statement with a conclusion referring to congruent angles such as, "If two angles have the same measure, then they are congruent." (Definition of Congruent Angles.)

Review the difference between Postulates and Theorems. It is important to note that the proofs of theorems are just like any other proof we write.

Paragraph proof – A type of proof written in paragraph form.

Example of Paragraph proof:

Given that $\overleftrightarrow{AC}$ intersects $\overleftrightarrow{CD}$ write a paragraph proof to show that A, C, D determine a plane.

Given: $\overleftrightarrow{AC}$ intersects $\overleftrightarrow{CD}$

Prove: A, C, D determine a plane.

$\overleftrightarrow{AC}$ and $\overleftrightarrow{CD}$ must intersect at C because if two lines intersect, then their intersection is exactly one point. Point A is on $\overleftrightarrow{AC}$ and point D is on $\overleftrightarrow{CD}$. Points A, C, D are not collinear. Therefore, A, C, D is a plane as it contains three points not on the same line.

Flow proof – A type of proof that uses arrows to show the flow of logical argument. Statements are connected by arrows to show how each statement comes from the one before it, and each reason is written below the statement it justifies.

***Preview topic** – Indirect proof (objective in Unit 4b)*

Indirect proof – a proof in which you prove that a statement is true by first assuming that its opposite is true. If the assumption leads to an impossibility (contradiction), then you have proved that the original statement is true.

Guidelines:

a. Identify the statement you are proving.
b. Assume its opposite to be true.
c. Continue writing a proof with statements that follow from the assumption.
d. If a contradiction is found, the assumption is **WRONG**!

Example: Use indirect reasoning to prove your position:

You claim that my dog dug up your garden. If we assume that was true, then he would have had to have been in your yard when the garden was being destroyed. I have a bill for the vet that says he was being groomed all day on the day your garden was being dug up.

Therefore, my dog DID NOT dig up your garden!

This is the kind of logic presented by defense attorneys when they provide their client's alibis in court.

Types of Proofs

- 2-Column proof
- Paragraph proof
- Flow proof
- Indirect proof (covered in Unit 4b)

A geometric proof could consist of five parts:

1) Diagram of the problem. Sometimes this is provided, if not it should be generated by using the given information and/or implication.
2) Given information. This is information that we start with. It may be written or notated on the diagram.
3) Proof statement. This is the object of the proof, the piece of information we try to prove exists.
4) Body of the proof. This is a sequence of statements which lead us to arrive at the proof statement. These statements can be presented in two columns, in a paragraph or in the form a flow chart.
5) Conclusion. The result of any completed proof must match the proof statement. The conclusion can be written as a conditional statement: the given information forms the hypothesis and the proof statement forms the conclusion.