Got Math?

A Newsletter from the Secondary Mathematics Team

Southern Nevada Regional Professional Development Program

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Factoring $ax^2 + bx + c$ - Teacher Notes

Here we look at sample teacher notes to develop instruction on how to factor the dreaded quadratic expression or equation after having taught multiplication of binomials. Sometimes this method is referred to as "splitting the middle term". Teachers should supplement with more examples as needed. Page 4 provides additional practice examples. Page 5 gives lyrics for a rap, song, or poem to learn the "ac method". Pages 6-8 of this newsletter provide background for the X-Games that can be used to reinforce the concept of looking for factors of a number that add to another number when working with quadratics in the form $x^2 + bx + c$.

Do you remember when we introduced the distributive property?
We may have explained that $2x + 3x$ could be written as $5x$, because $(2+3)x$ OR $x(2+3)$ allows us to *combine like terms* (in this case constants), 2 and 3.

And, in the reverse order, that $5x$ could also be written as $2x+3x$. It didn't seem like earth-shattering news for such simple examples, but sometimes students missed the concept completely. Maybe because it was too easy.

Later, we used the distributive property for factoring. For example, $x^2 + 2x$ became $x(x+2)$, $6x+12$ factored to $6(x+2)$, $-3x+6$ to $-3(x-2)$, and so on. It's interesting that one of the first steps we teach in factoring is to check for common factors, but that step often gets missed by students when we teach factoring quadratics.

Example 1: $x^2 + 2x + 3x + 6$

$x^2 + 2x + 3x + 6$ can be grouped, $(x^2 + 2x) + (3x + 6)$ and then factored using the distributive property on each piece, as $x(x+2) + 3(x+2)$.
Hopefully we notice that there is still a common factor, $(x+2)$ that can be factored out.

So, $x^2 + 2x + 3x + 6 = (x^2 + 2x) + (3x + 6) = = x(x+2) + 3(x+2)$ OR $(x+3)(x+2)$.

Example 2: $x^2 + \underline{3x + 4x} + 12$ Review the patterns to combine the like terms or write them differently.

The middle term $7x$ can be "split" into two terms. This reinforces the link with the inverse operation of adding or combining like terms. We then factor by grouping.
Yes, teachers ALWAYS do problems showing how to combine like terms, but do we develop possible ways to combine terms to achieve a given sum and product?

$x^2 + \underline{3x + 4x} + 12$ can be written as $x^2 + \underline{7x} + 12$; AND $x^2 + \underline{7x} + 12$ can be written in place of $x^2 + \underline{3x + 4x} + 12$.

Example 3: $x^2 + 5x + 4x + 20$

Notice that this example is written as $x^2 + 5x + 4x + 20$ instead of $x^2 + 9x + 20$.

Why do this? We know that when we teach factoring and it is seen as $x^2 + 9x + 20$, we ALWAYS say, "we are looking for factors of 20 whose sum is 9". Then we often jump to trial and error and list binomial factors $(x + \quad)(x + \quad)$. Maybe we list all factors and "try" 1 and 20, or 2 and 10, or 4 and 5. If we are lucky enough to choose 5 and 4, we place them in the parentheses in the correct place and check by multiplying the binomials.

We see with multiplication that $(x + 5)(x + 4) = x^2 + 5x + 4x + 20$ and simplify or combine like terms to get $x^2 + 9x + 20$.
It's pretty interesting that when the correct factors are chosen AND written into the expression as $x^2 + 5x + 4x + 20$, the middle term has been "split" into two parts that add to $9x$ and the coefficients multiply to 20.

Now, to finish the example, $\underline{x^2 + 5x} + \underline{4x + 20}$ is grouped as underlined, or placed in parentheses $(x^2 + 5x) + (4x + 20)$, then factored using the distributive property again,
$x(x+5) + 4(x+5)$.

AND, factoring out the common factor $(x+5)$ we get $(x+5)(x+4)$. Yeh!!!

Math Resources

www.rpdp.net

In this issue:

Factoring $ax^2 + bx + c$ - Teacher Notes continued

What about the student who mixes up the 4 and the 5 and writes $\underline{x^2 + 4x} + \underline{5x + 20}$?
This results in $x(x+4) + 5(x+4)$, then, finally the same two factors, $(x+4)(x+5)$. Interesting, huh?

Will this always work?

When you can split the middle term in such a way that the product of the two "like term" coefficients is the last term (constant), the trinomial IS factorable. What's more, if the middle term can not be split it in such a way that the product of the two "like term" coefficients is the last term (constant), the trinomial is NOT factorable. (This also works for difference of squares.)

Examples 4 – 7: These examples show importance of signs.

$\mathbf{x^2 + 8x + 12}$	$\mathbf{x^2 - x - 12}$	$\mathbf{x^2 + 10x - 24}$	$\mathbf{x^2 - 10x + 24}$
$x^2 + 6x + 2x + 12$	$x^2 + 3x - 4x - 12$	$x^2 + 12x - 2x - 24$	$x^2 - 6x - 4x + 24$

Yes, students can often "just do it", but keep following the connections to see patterns and practice the skill development. Try factoring each of these examples. Be careful with the signs.

Example 8: $\mathbf{2x^2 + 11x + 12}$ **OR** $\mathbf{ax^2 + bx + c}$

How about $2x^2 + 11x + 12$? What's different? Let's look at the method sometimes referred to as the **"ac method"**, which is just an extension of the method describe above, splitting the middle term.

Looking back at the simple examples, when a = 1, we factored c. Everyone knows that ac=c, when a=1.
After identifying 'a' and 'c' we multiply a times c. Now we factor ac, or in this case 2(12) or 24, and look for factors of 24 that add to 11.

It is easy to show that $3 + 8 = 11$, so $3x + 8x = 11x$.
We can write $2x^2 + 11x + 12$ as $2x^2 \underline{+ 3x + 8x} + 12$,

then group (associate) $\underline{2x^2 + 3x} + \underline{8x + 12}$.
Factor each group $x(2x+3) + 4(2x+3)$ by factoring out the common factor $(2x+3)$.
The factors are $(x+4)(2x+3)$.

So, $2x^2 + 11x + 12 = (x+4)(2x+3)$. Check by FOIL and multiplication of means-extremes.

More examples? Variations?

Example 9: $\mathbf{12x^2 - 11x - 15}$

$a(c) = 12\ (-15) = -180$
Factors of -180 that differ by 11
. . . . .9 & 20, specifically 9 and -20

So, $12x^2 - 11x - 15$ can be written as

$12x^2 \underline{- 20x + 9x} - 15$ OR $12x^2 + \underline{9x - 20x} - 15$. Take your pick!

Group $\underline{12x^2 - 20x} + \underline{9x - 15}$	OR	$\underline{12x^2 + 9x} - \underline{20x - 15}$
$(12x^2 - 20x) + (9x - 15)$		$(12x^2 + 9x) + (-20x + 15)$
$4x(3x - 5) + 3(3x - 5)$		$3x(4x+3) - 5(4x+3)$
$(4x+3)(3x - 5)$	OR	$(3x - 5)(4x+3)$

Therefore, $12x^2 - 11x - 15 = (4x+3)(3x - 5)$. Check by multiplication.

Let's try another example with a slight variation.

Example 10: $\mathbf{4x^2 + 4x - 24}$ Note that there is a common factor that could/should be factored out of each term first.

$4x^2 + 4x - 24$ can be written as
$4(x^2 + x - 6)$, and it fits into our plan for factoring without the 'big' numbers. Keep the 4 as a factor.
Then, splitting the middle term, since $+x = 3x - 2x$,
$4[x^2 + 3x - 2x - 6]$ which, when grouped looks like
$4[(x^2 + 3x) - (2x + 6)]$ and $4[x(x + 3) - 2(x + 3)]$.

So $4x^2 + 4x - 24 = 4[(x + 3)(x - 2)]$.

Factoring $ax^2 + bx + c$ - Teacher Notes continued

Example 11: $\mathbf{6x^2 + 2x - 20}$

Again, there is a common factor 2, so it becomes $2(3x^2 + x - 10)$. Use 'a' = 3 and 'c' = – 10, and we keep the factor 2 throughout the problem.

$ac = 3(-10) = -30$
Factors of – 30 that combine to +1
. . . . .– 5 & 6, because $(-5 + 6 = 1)$

So, $2(3x^2 + x - 10)$ can be written as

$2(3x^2 + 6x - 5x - 10)$ OR $2(3x^2 - 5x + 6x - 10)$. Take your pick!

Group $2(\underline{3x^2 + 6x} - \underline{5x - 10})$ OR $2(\underline{3x^2 - 5x} + \underline{6x - 10})$

And factor out the common factors

$2[3x(x+2) - 5(x + 2)]$ $\qquad$ $2[x(3x - 5) + 2(3x - 5)]$

So, $6x^2 + 2x - 20 = 2(3x - 5)(x + 2)$ OR $2(x + 2)(3x - 5)$

Can it be used to identify trinomials that do not factor?

Example 12: **Does $3x^2 - 20x + 7$ factor?**
$ac = 3(7) = +21$

Factors of +21 are 1, 21; – 1, – 21; 3, 7; or – 3, – 7

Since factors must be the same sign to give a product of positive 21, there are no factors of 21 with the same sign that combine to be 20.

Therefore, $3x^2 - 20x + 7$ is not factorable. (You might show that $3x^2 - 20x - 7$ is factorable. Why?)

How about difference of squares? Trinomial squares?

Example 13: $\mathbf{x^2 - 64.}$

Same plan: factor –64 so the factors combine to 0. (8 and – 8)

It is easily shown that $x^2 + 8x - 8x - 64$ factors to $x(x+8) - 8(x+8)$ and then $(x - 8)(x+8)$.

$x^2 - 64 = (x-8)(x+8)$.

<u>**Summary:**</u>

The connections and linkage to previous instruction are numerous. Throughout this technique, teachers can refer to concepts such as factoring integers, distributive property and factoring, sum of factors, and product of factors. The method has benefits that link writing math expressions for the words we use to teach factoring - "factors that multiply to be the constant and combine to the middle term". The method may save time from trial and error; it finds use/practice for the distributive property that is important in future math applications; it stresses the connection to sum and product of roots, listing factors in a factor tree, and it aids in determining if a trinomial is factorable. There are also links to factoring with multiplying the means and extremes of binomials to get the middle term.

Factoring quadratics in the form $x^2 + bx + c$ can be accomplished by a method referred to as "splitting the middle term". This method can also be extended for quadratics in the form $ax^2 + bx + c$ by using the "ac method".

The algorithm for the "ac method" is as follows:

1. **Form the product "ac".**
2. **Find a pair of numbers b_1 and b_2 whose product is also "ac" and whose sum is "b" (if such integers don't exist, it can't be factored using rational numbers).**
3. **Split the middle term using b_1 and b_2; that is, express the term bx as $b_1x + b_2x$.**
4. **Now you have 4 terms and you will be factoring by grouping pairs of terms.**
5. **Factor out the common factor in each pair of terms. The common factor at this point should be a binomial.**

Check by multiplying. Notice the means-extremes products should be your "split".

Factoring $ax^2 + bx + c$ - Example Problems

ExampleProblems
Use “split the middle term” or “ac method”

Factor each example, if possible! Write “not-factorable” if it can’t be factored.

1	$x^2 - 5x + 2x - 10$	2	$x^2 - 7x + 6$
3	$x^2 - 8x + 16$	4	$3x^2 + 4x + 2$
5	$10x^2 + 13x - 3$	6	$8x^2 + 12x + 3$
7	$2x^2 + \_\_\_\_ x + 15$ Fill in the blank with a number that makes the trinomial factorable. How many ways can it be done?	8	$x^2 + 12x + 36$
9	$2x^2 - 11x - 6$	10	$4x^2 - 9 = 0$
11	$4x^2 - 64 = 0$ Be careful	12	$3x^2 + 11x - 4 = 0$
13	$8x^2 - 22x + 5 = 0$	14	$3x^2 + 9x - 12 = 0$ Be careful

The Factor Rap

The Factor Rap (song, poem, ditty…)

Common Factors ← First line: said enthusiastically.

Example

$ax^2 + bx + c$

$2x^2 + 11x + 12$

$2*12 = 24$

To factor quadratic trinomials, find *a* times *c*

Then factor out the product so the sum is *b*

$I\ use\ \ 24 = 8*3$

$because\ \ 8+3 = 11$

$2x^2 + 8x + 3x + 12$

We write the thing again so the middle term is split

$(2x^2 + 8x) + (3x + 12)$

Then we pair the terms we see to make groups out of it

$2x(x+4) + 3(x+4)$

Factor common factors with distributive property

$(2x+3)(x+4)$

Now we do the same again . . It's as easy as can be.

Finale. . .

But wait, if the sum of the factors can never equal *b*

The trinomial doesn't factor, it's as easy as can be!

X-Games, X-Patterns, X-Factors, Diamonds

This activity can be used from the beginning of the school year to get students more familiar and comfortable with patterns. It provides very good practice to review material you want to make sure students remember or to preview something they will see later on. It should be a short (time) activity that stimulates student engagement. X-Games show patterns for multiplying, then extend the patterns to factoring and looking for specific factors.

Getting started:
Begin by drawing an X on the board. Refer to the openings as top, bottom, right, and left.

Write numbers in the right and left and ask students to guess what goes in top and bottom. To teach the game, place the sum of right and left in the top opening. The bottom will be the product of right and left. Although it is not critical whether the product is placed in the top or bottom of the X, it is important to maintain consistency throughout the year. However, you need not tell the students these rules. In the beginning you want them to feel free to guess and not to feel like they are expected to know the right answers. Try to find a positive way to respond to each wrong answer.

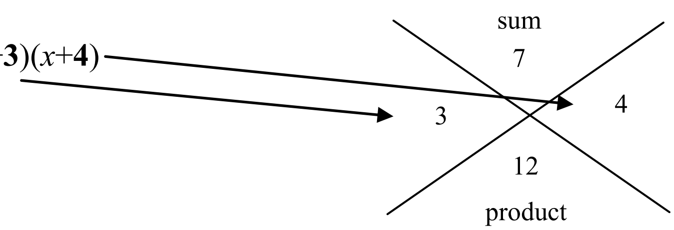
Provide examples and give hints, as needed, until students come up with the right answers. Write those answers in the appropriate places. Some students will begin to see the pattern very quickly.

Once the students seem to have discovered the pattern, have a student explain it to the class. Then start varying the given information by filling in top and right, bottom and left, top and bottom, etc. Also, now that students know the pattern, you can also start using negatives, decimals, fractions, variable expressions, or any other expressions that you want to review or preview in a non-threatening, fun way.

As a variation, you may also decide to change the way the top and bottom numbers are determined. For example, top might be the result of squaring the number in the left and adding the right number, while bottom might be found by doubling the number in the left and adding the right number. This would demonstrate contrast between squaring and doubling.

Example:
There is a special pattern for multiplying and/or factoring trinomials.
For example, to multiply $(x+3)(x+4)$ put the 3 in the right (or left) and the 4 in the left (or right).
Then fill in the top and bottom. The top will be the sum and bottom will be the product.

The product would be pulled from the X as $(x+\mathbf{3})(x+\mathbf{4})$



The product is $\boldsymbol{x^2 + 7x + 12}$.

To factor $\boldsymbol{x^2 + 7x + 12}$, do the reverse. That is, put the 7 at top, 12 at bottom, then find the two numbers whose sum is 7 and whose product is 12. You get $(x+\mathbf{3})(x+\mathbf{4})$.

Again, the X-Games can be used for other math practice. Practice using the X concept with fractions, integers, radicals, absolute value, opposites, factorials, squares, cubes, double and add , and vocabulary.

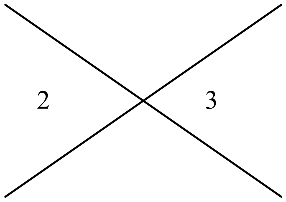
Patterns for Multiplying Binomials

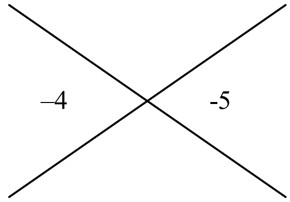
[Same signs]

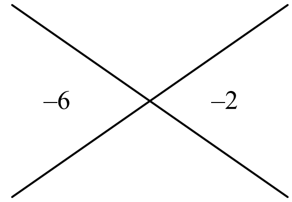
$(x + 2)(x + 3) =$

$(x - 4)(x - 5) =$

$(x - 6)(x - 2) =$





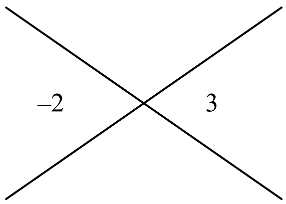


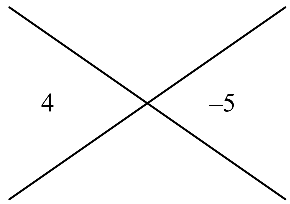
[Different signs]

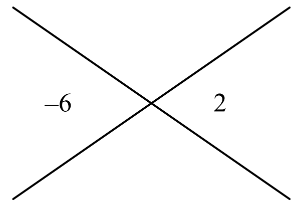
$(x - 2)(x + 3) =$

$(x + 4)(x - 5) =$

$(x - 6)(x + 2) =$





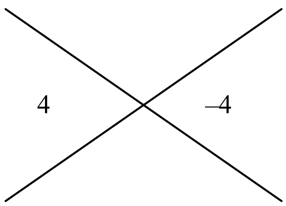


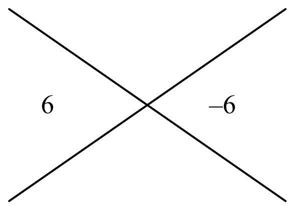
[Conjugates]

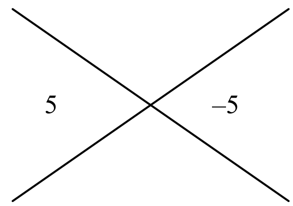
$(x + 4)(x - 4) =$

$(x + 6)(x - 6) =$

$(x - 5)(x + 5) =$





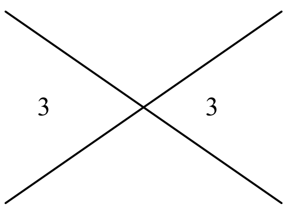


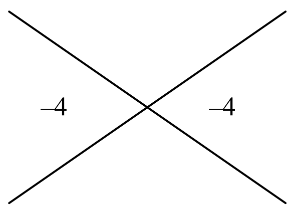
[Binomial squared]

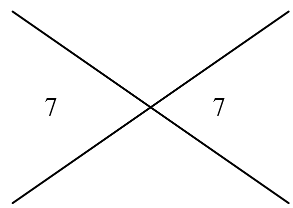
$(x + 3)^2 =$

$(x - 4)^2 =$

$(x + 7)^2 =$







You can get through a lot of examples quickly with the X-Factor.
The pattern is adding to get the middle term and multiplying to get the last term, **every time**.

Note: This is not to replace instruction, just to provide quick mental connections. <u>ALWAYS bring them back to the actual multiplication methods utilizing the distributive property.</u>

Now: What if we UNDO the multiplication process (factor)?

NEXT: It is interesting to note that $(x + 2)(x + 3) = x^2 + \underline{2x + 3x} + 6 = x^2 + \underline{5x} + 6$

Notice that, going backwards, $5x$ is split up into $\underline{2x + 3x}$

Does this always work for these kinds of problems?

Patterns for Factoring Trinomials

[Same signs]

$(x + \_\_)(x + \_\_) =$

$(x - \_\_)(x - \_\_) =$

$(x \_\_\_)(x \_\_\_) =$

$x^2 + 5x + 6$	$x^2 - 9x + 20$	$x^2 - 8x + 12$
top: 5	top: –9	top: –8
bottom: 6	bottom: 20	bottom: 12

[Different signs]

$(x - \_\_)(x + \_\_) =$

$(x + \_\_)(x - \_\_) =$

$(x \quad \_\_)(x \quad \_\_) =$

$x^2 + x - 6$	$x^2 - x - 20$	$x^2 - 4x - 12$
top: 1	top: –1	top: –4
bottom: –6	bottom: –20	bottom: –12

[Conjugates]

$(x + \_\_)(x - \_\_) =$

$(x - \_\_)(x + \_\_) =$

$(x \quad \_\_)(x \quad \_\_) =$

$x^2 - 16$	$x^2 - 36$	$x^2 - 25$
top: 0	top: 0	top: 0
bottom: –16	bottom: –36	bottom: –25

[Binomial squared]

$(x + \_\_)^2 =$

$(x - \_\_)^2 =$

$(x \quad \_\_)^2 =$

$x^2 + 6x + 9$	$x^2 - 8x + 16$	$x^2 + 14x + 49$
top: 6	top: –8	top: 14
bottom: 9	bottom: 16	bottom: 49

After the pattern is well developed, discuss with students what they could use for the other three numbers if only the top number was provided.

You can get through a lot of examples quickly with the X-Factor.
The pattern is figuring out what to add to get the middle term and also multiply to get the last term.

Note: This is not to replace instruction, just to provide quick mental connections. ALWAYS bring them back to the actual multiplication methods utilizing the distributive property and the inverse of factoring.